Adjoint Operators

Thursday, July 9, 2020 10:17 AM

Definition

Let $f: U \longrightarrow V$. The adjoint of f, f^*, is a map $f^*: V \longrightarrow U$ such that

$V \ni \langle f(u), v \rangle = \langle u, f^*(v) \rangle \in U$

Facts:

i. $(\alpha f + g)^* = \bar{\alpha} f^* + g^*$

ii. $(fg)^* = g^* f^*$

iii. $(f^*)^* = f$

iv. $I^* = I$

v. $A^*_{ij} = \overline{A_{ji}}$ for a matrix $A \in \mathbb{C}^{m \times n}$

☆ A matrix/operator is self-adjoint / Hermitian if it equals its adjoint

Spectral Theorem

A linear map $f: V \longrightarrow V$ is self-adjoint iff it is real and diagonal in an orthonormal basis.

Proof:

1 Spectral Theorem

For a complex $n \times n$ Hermitian matrix A,

a) All eigenvalues of A are real.

b) A has n linearly independent eigenvectors $\in \mathbb{C}^n$.

c) A has orthogonal eigenvectors, i.e., $A = V\Lambda V^{-1} = V\Lambda V^*$, where Λ is a diagonal matrix and V is a unitary matrix. We say that A is orthogonally diagonalizable.

Recall that a matrix A is Hermitian if $A = A^*$. Furthermore, if A is of the form B^*B for some arbitrary matrix B, all of its eigenvalues are non-negative, i.e., $\lambda \geq 0$.

a) Prove the following: All eigenvalues of a Hermitian matrix A are real.

Hint: Let $(\lambda, \vec{v})$ be an eigenvalue/vector pair and use the definition of an eigenvalue to show that $\lambda\langle \vec{v}, \vec{v}\rangle = \bar{\lambda}\langle \vec{v}, \vec{v}\rangle$.

Let $\lambda \in \mathbb{R}$ be an eigenvalue of A of $V \in \mathbb{C}^n$.

Let $\lambda \in \mathbb{R}$ be an eigenvalue of A of $V \in \mathbb{C}^n$.
Such that $Av = \lambda v$.

inner products are linear w.r.t. first input

Then, $\langle Av, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle$.
By definition of adjoint, $\langle Av, v \rangle = \langle v, A^* v \rangle$.
Since A is a Hermitian matrix, $A^* = A$ so $\langle v, A^* v \rangle = \langle v, Av \rangle = \langle v, \lambda v \rangle$.
$\langle v, \lambda v \rangle = \bar{\lambda} \langle v, v \rangle$.
Therefore $\lambda \langle v, v \rangle = \bar{\lambda} \langle v, v \rangle$ so $\lambda = \bar{\lambda}$ and λ must be real.

b) Prove the following: For any Hermitian matrix A, any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal.
Hint: Use the definition of an eigenvalue to show that $\lambda_1 \langle \vec{v}_1, \vec{v}_2 \rangle = \lambda_2 \langle \vec{v}_1, \vec{v}_2 \rangle$.

Goal: Show $\langle v_1, v_2 \rangle = 0$.

same as a)

$\langle A v_1, v_2 \rangle = \langle \lambda_1 v_1, v_2 \rangle = \lambda_1 \langle v_1, v_2 \rangle$ ✓

$\langle A v_1, v_2 \rangle = \langle v_1, A^* v_2 \rangle$ (by adjoint)
$= \langle v_1, A v_2 \rangle$ (by Hermitian)
$= \langle v_1, \lambda_2 v_2 \rangle$ (by eigenvalue)
$= \bar{\lambda}_2 \langle v_1, v_2 \rangle$
$= \lambda_2 \langle v_1, v_2 \rangle$ (since λ_2 is real by a)

Therefore, $\lambda_1 \langle v_1, v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle$.
Since λ_1, λ_2 are distinct, $\langle v_1, v_2 \rangle$ must be 0 and thus v_1 and v_2 are orthogonal.

c) Prove the following: For any matrix A, A^*A is Hermitian and only has non-negative eigenvalues.

i. A^*A is Hermitian.

Let A be an arbitrary matrix $\in \mathbb{C}^{m \times n}$.
Verify $A^*A = (A^*A)^*$

Know that by the reverse decomposition property,
$(fg)^* = g^* f^*$, so

$$A^*A = A^*(A^*)^* = A^*A.$$

ii. $\lambda \geq 0$ for all eigenvalues of A^*A.

Start with $v^*(A^*A)v = \langle (A^*A)v, v \rangle$.

$$(v^*A^*)(Av)$$
$$= (Av)^* Av \quad \text{(reverse decomposition)}$$
$$= \langle Av, Av \rangle.$$

Now, start with the equivalent $v^* \lambda v = \lambda v^* v$.
$$= \lambda \langle v, v \rangle.$$

Therefore $\lambda \langle v, v \rangle = \langle Av, Av \rangle$.

must be ≥ 0 because v is an eigenvector of A and must be positive

Restriction

Let $f : V \longrightarrow V$ be a linear map.
If $W \subseteq V$ and $f(W) \subseteq W$, then W is "f-stable" or "f-invariant"

Example: shear  stabilizes the x-axis

If W is f-invariant, the restriction of f to W
$f|_W : W \longrightarrow W$ is equal to $f(W) = W$

Application: max amplification of a matrix A

What is the maximum value $\dfrac{\|Av\|}{\|v\|}$?

If $\|v\| = 1$: $\max_v \|Av\| = \max_v \sqrt{\langle Av, Av \rangle}$ Use $\langle v, Av \rangle = \langle A^*v, v \rangle$
$$= \max_v \sqrt{\langle A^*Av, v \rangle}$$

Diagonalize A^*A to get $= \max_v \sqrt{\langle U \Lambda U^* v, v \rangle}$
$$= \max_v \sqrt{\langle \Lambda U^* v, U^* v \rangle}$$

1) Diagonalize A^*A to get $= \max_{,} \sqrt{\langle U \Lambda U^* v, v \rangle}$

$= \max_{,} \sqrt{\langle \Lambda U^* v, U^* v \rangle}$

We know Λ is a diagonal matrix of eigenvalues, so choose λ_{max} from Λ to substitute.

$$\sigma_1 = \sqrt{\lambda_{max}(A^*A)}$$