# Controllability of Discrete Time Systems

## Definitions of controllability:

• Choose any $x_{goal} \in X = \mathbb{R}^{n}$ ⇒ $n$ = number of state variables

A system is controllable if there exist a sequence of controls $u_0, u_1 \dots u_{k-1}$ that bring a system from $x_0$ to $x_k = x_{goal}$.

## Mathematical definition:

$$x_k - A^k x_0 = \underbrace{\begin{bmatrix} B & AB & \dots & A^{k-1}B \end{bmatrix}}_{\text{controllability matrix}} \begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix}$$

A system is controllable if this matrix is full rank (column space $= \mathbb{R}^n$).
↑ all vectors linearly independent

⇐ Cayley-Hamilton Theorem: MAX steps to get to any state in controllable system is $n$

* if a system is controllable at $x_n$, it is also controllable at $x_k$ for $k > n$. (because of physical limits we usually take more than $n$)

**Proof**: if $\text{rank}(\underbrace{[b \; Ab \dots A^{\ell-1}b]}_{n \times \ell \text{ matrix}}) = \ell$

$\text{rank}([b \; Ab \dots A^{\ell}b]) == \ell$, then

$\text{rank}([b \dots A^{k-1}]) == \ell$ for $k \geq \ell$.

↳ span $= \mathbb{R}^\ell$ so vectors are linearly independent.

Proof by induction:

Base case: if $\text{rank}(\underbrace{[b \; Ab \dots A^{\ell}b]}_{n \times (\ell+1)}) = \ell$ then

$\text{rank}([b \; Ab \dots A^{\ell-1}b]) = \ell$ since the last vector must be linearly dependent.

Inductive Hypothesis: for $k \geq \ell$, then

$$A^k b = \alpha_0^k b + \dots + \alpha_{k-1}^k A^{\ell-1} b \quad \text{for any } k \in \mathbb{N}.$$

$$A^{k+1} b = A(A^k b) = \alpha_0^k A b \dots + \alpha_{k-1}^k A^{\ell-1} b$$

$$= \alpha_{k-1}^k \alpha_0^k b + (\alpha_{k-1}^k \alpha_1^k + \alpha_0^k) A b + \dots +$$

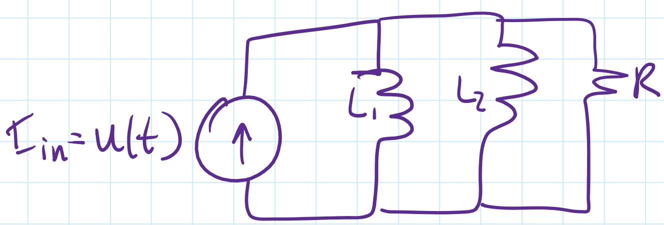
$$(\alpha_{k-1}^{k\,2} + \alpha_{k-2}^k) A^{\ell-1} b.$$

This shows that for all $k > \ell$, adding 1 to $k$ keeps the rank the same.

Therefore, controllability is equivalent to $\text{rank}[b \ Ab \cdots A^{n-1} b] = n$ since any further additions to $k$ will not allow the system to increase in rank.

Multiple inputs: $C = [\underbrace{B \quad AB \quad \cdots \quad A^{n-1} B}_{n \times (n \cdot m)}]$

Example:




$$\bullet \ L_1 \frac{d i_1}{dt} = V_1 = V_2 = V_R = I_R R = R\big(u(t) - i_1(t) - i_2(t)\big)$$

Define $x(t) = \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}$ to get the system

$$\dot{x}(t) = \begin{bmatrix} \dot{i}_1 \\ \dot{i}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{R}{L_1} & -R/L_1 \\ -R/L_2 & -R/L_2 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} R/L_1 \\ R/L_2 \end{bmatrix}}_{B} u(t)$$

$$Ab = \begin{bmatrix} -\frac{R}{L_1}\left(\frac{R}{L_1} + \frac{R}{L_2}\right) \\ -\frac{R}{L_2}\left(\frac{R}{L_1} + \frac{R}{L_2}\right) \end{bmatrix} = -\left(\frac{R}{L_1} + \frac{R}{L_2}\right) b$$

$[b \quad Ab]$ is not full rank since $Ab$ is a multiple of $b$!

↳ not a controllable system

## 2 Deadbeat Control

Consider the system

$$\vec{x}[t+1] = A\vec{x}[t] + Bu[t] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[t].$$

a) Is this system controllable?

*Check if $\det(C) = 0$

$$C = [B \quad AB] = \begin{bmatrix} 0 & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 & \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$\det\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = 1 \neq 0$ so this system is controllable.

b) For which initial states $\vec{x}[0]$ is there a control that will bring the state to zero in a single time step?

*Find $\vec{x}[0] = \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix}$ such that the final state is $x[1] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$x_1 = Ax_0 + Bu_0$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_{1_0} \\ x_{2_0} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_0$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1[0] - x_2[0] \\ x_2[0] - x_1[0] + u_0 \end{bmatrix}$$

← not controllable so $x_1[0] - x_2[0]$ must equal 0.

← controllable (in 1 step)

c) For which initial states $\vec{x}[0]$ is there a control that will bring the state to zero in two time steps?

$$x[2] = Ax[1] + Bu[1] = A(Ax[0] + Bu[0]) + Bu[1]$$

$$= A^2 x[0] + ABu[0] + Bu[1]$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x_1[0] - 2x_2[0] - u[0] \\ 2x_2[0] - 2x_1[0] + u[0] + u[1] \end{bmatrix}$$

both controllable

so all initial states can be brought to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in 2 time steps.

d) Now let $\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ be the initial state. Give a set of control inputs $u[0]$ and $u[1]$ to bring to system to $\vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in two time steps.

From c),

$$\begin{bmatrix} \phantom{0} \end{bmatrix} = \begin{bmatrix} 2x_1[0] - 2x_2[0] - u[0] \end{bmatrix}$$

From c),

$$y = x[2] = \begin{bmatrix} 2x_1[0] - 2x_2[0] - u[0] \\ 2x_2[0] - 2x_1[0] + u[0] + u[1] \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 - u[0] \\ -2 + u[0] + u[1] \end{bmatrix}$$

$u[0] = 2$

$$1 = -2 + 2 + u[1]$$

$u[1] = 1$

From scratch:

$$C \begin{bmatrix} u[1] \\ u[0] \end{bmatrix} = x[2] - A^2 x[0]$$

$$\begin{bmatrix} u[1] \\ u[0] \end{bmatrix} = C^{-1}\left(x[2] - A^2 x(0)\right)$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

# SVD for Least Squares

Goal: Find $y = Ax + e$ to minimize the error $e$.
↳ equivalent to finding $\min \|Ax - y\|$

$$m\ \#\text{rows}\left\{ \overbrace{\begin{bmatrix} \quad & \quad \\ & \\ & \end{bmatrix}}^{n\ \#\text{cols}} \right.$$

Substitute SVD of $A$ to get

$$\min \| U \Sigma V^T x - y \|$$

Factor out $U$ to get

$$\min \| U(\Sigma V^T x - U^T y \|$$

Since $U$ has norm 1, this is equal to

$$\min \| \Sigma V^T x - U^T y \|$$

Let $\Sigma = \begin{bmatrix} S & 0 \\ 0 & \ddots \; 0 \end{bmatrix}$ where $S$ is the diagonal matrix of singular values.

$$\min \| SV^T x - U^T y \| = \min \left\| \begin{bmatrix} SV^T x & -U_1^T y \\ 0 & \vdots \\ & -U_n^T y \end{bmatrix} \right\|$$

$$= \min \| SV^T x - U_1^T y \|^2 + \dots + \| -U_n^T y \|^2$$

If we choose an $x$ to minimize, we can't influence the last term which has no $x$.

Since $S$ is diagonal and $V^T$ exists, $S$ and $V$ are invertible so let

$$SV^T x - U_1^T y = 0$$

$$SV^T x = U_1^T y$$

$$x^* = VS^{-1} U_1^T y = (A^T A)^{-1} A^T y \text{ when } A \text{ is full rank}$$

# Undetermined Case

Problem: minimize (for an orthogonal matrix $V$)

$z_1 \in \mathbb{R}^n$ and $z_2 \in \mathbb{R}^{m-n}$ for

$\|V_1 z_1 + V_2 z_2\|_2^2$ such that

$y = AV_1 z_1 + AV_2 z_2$

$$= \left(\langle V_1 z_1 + V_2 z_2, V_1 z_1 + V_2 z_2 \rangle\right)^2$$

$$= (V_1 z_1 + V_2 z_2)^* (V_1 z_1 + V_2 z_2)$$

$$= V_1^* z_1^* V_1 z_1 + \underline{V_1^* z_1^* V_2 z_2 + V_2^* z_2^* V_1 z_1} + V_2^* z_2^* V_2 z_2$$

both equal zero since $V_1$ and $V_2$ are orthogonal

$$= \|V_1 z_1\|^2 + \|V_2 z_2\|^2$$

$AV_2$ is $0$ since $V_2 \in \mathrm{Null}(A)$.

$$AV_2 = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} V_2$$

$$= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S & 0 \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix} = 0$$

$\underset{z_1, z_2}{\text{minimize}}\ \|V_1 z_1\|_2^2 + \|V_2 z_2\|^2$ such that $y = AV_1 z_1$.

★ $z_2$ must be the zero vector for this to be true. So

$$y = AV_1 z_1 = U \begin{bmatrix} S & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} V_1 z_1$$

$$= U \begin{bmatrix} S & 0 \end{bmatrix} \begin{bmatrix} I \end{bmatrix} \ldots$$

$$y = AV_1 z_1 = U[S \;\; 0]\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} V_1 z_1$$

$$= U[S \;\; 0]\begin{bmatrix} I \\ 0 \end{bmatrix} z_1$$

$$y = \underset{\substack{n \times n \\ \text{Orthonormal}}}{U} \; \underset{\substack{n \times n \\ \text{diagonal}}}{S} \; \underset{n \times 1}{z_1}$$

$$\boxed{z_1^* = S^{-1} U^T y}$$

## Summary

When $A$ is full column rank,

$$A^+ = (A^T A)^{-1} A^T$$

When $A$ is full row rank,

$$A^+ = (A^T (AA^T)^{-1})$$

Pseudo-inverse of $A$

# Optimal Control

Consider $x_{k+1} = Ax_k + Bu_k$ for $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$

The controllability matrix

$$C_k = [B \;\; AB \;\; \ldots \;\; A^{n-1}B]$$

If the system is controllable:

We can minimize $\left\| \begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix} \right\|$ such that

$$(x_{goal} - A^k x_0) = C_k \underbrace{\begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix}}_{k \text{ by } m}$$

$$(x_{goal} - A^k x_0) = C_k \begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix} \Big\} k \text{ by } m$$

Solve by using the least squares solution:

$$\begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix} = C_k^T (C_k C_k^T)^{-1} (x_{goal} - A_k x_0).$$

Not every control is weighted the same -

## Car Example

System: $$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{RM} \end{bmatrix} u(t)$$

Assume zero order hold:

$$u(t) = u_k : t \in [kT, (k+1)T)$$

Then $$\begin{cases} v(t) = v(kT) + \int_{kT}^{t} \frac{1}{RM} u_k = V_k + \frac{(t-kT)}{RM} u_k \\ p(t) = p(kT) + \int_{kT}^{t} \left( V_k + \frac{(s-kT)}{RM} \right) ds \end{cases}$$

$$v((k+1)T) = V_k + \frac{T}{RM} u_k$$
$$p((k+1)T) = p_k + T V_k + \frac{T^2}{2RM} u_k$$

$$x_{k+1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_k + \frac{1}{RM} \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u_k$$

Optimal control: find $u$ such that $x_{goal} = x_k = \begin{bmatrix} p_{goal} \\ 0 \end{bmatrix}$.

- Check controllability: $C = [b \quad Ab]$
$$= \frac{1}{RM} \begin{bmatrix} \frac{T^2}{2} & \frac{3T^2}{2} \\ T & T \end{bmatrix} \Rightarrow \text{controllable}$$

$$\begin{bmatrix} u_1 \\ u_0 \end{bmatrix} = C_2^{-1} \begin{bmatrix} p_{goal} \\ 0 \end{bmatrix}$$

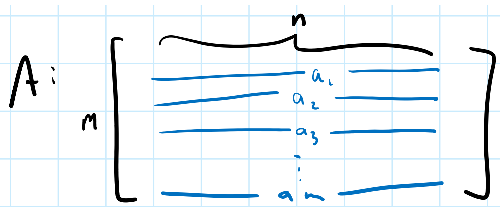
## 2 Minimum Energy Warmup

Consider an undetermined system of equations given by

$$\vec{y} = A\vec{x}$$

where $A \in \mathbb{R}^{m \times n}$ is a wide matrix, that is $m < n$. We assume that $A$ has linearly independent rows.

a) What is the rank of $A$?




rank = column space.

$\text{Rank}(A) = m$

b) In lecture, we saw that the minimum norm solution for $\vec{x}$ is given by

$$\vec{x} = V_c \Sigma_c^{-1} U_c^T \vec{y} = V_c \Sigma_c^{-1} U^T \vec{y}$$

where the definition of $V_c$ and $\Sigma_c$ come from the block matrix form of the SVD. That is,

$$A = U \Sigma V^T = U \begin{bmatrix} \Sigma_c & 0_{m \times (n-m)} \end{bmatrix} \begin{bmatrix} V_c^T \\ V_2^T \end{bmatrix} \qquad (3)$$

Argue that $\Sigma_c$ is invertible. What are the matrix elements $\Sigma_{c_{ij}}$ and $\Sigma_{c_{ij}}^{-1}$?

Since $U$ and $V$ are orthogonal, $\text{rank}(A) = \text{rank}(\Sigma)$.

$\Sigma_c = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \end{bmatrix}$ $(m \times m)$ so $\Sigma_c^{-1} = \begin{bmatrix} \frac{1}{\sigma_1} & & \\ & \ddots & \\ & & \frac{1}{\sigma_m} \end{bmatrix}$.

c) Use the SVD of $A$ to show that the expression for the minimum-norm solution from equation 3 can also be written as

$$\vec{x} = A^T (AA^T)^{-1} \vec{y}$$

HINT: $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$ for invertible matrices $A$, $B$, $C$.

→ Show $A^T (AA^T)^{-1} y = V_c \Sigma_c U^T y$

- Show $A^T(AA^T)^{-1}y = V_C\Sigma_C U^{-1} y$

Know $A = U_c\Sigma_c V_c^T$. Plug into left side:

$$(U_c\Sigma_c V_c^T)^T\left(U\Sigma_c V_c^T (U\Sigma_c V_c^T)^T\right)^{-1} y$$

$$= (V_c\Sigma_c^T U^T)(U\Sigma_c V_c^T V_c\Sigma_c U^T)^{-1} y$$

- $U, V$ are orthogonal matrices so $V^TV = I$.

$$= (V_c\Sigma_c^T U^T(U\Sigma_c^2 U^T)^{-1} y.$$

$$= V_c\Sigma_c^T U^T(U^{T^{-1}}\Sigma_c^{2^{-1}} U^{-1}) y$$

$$= V_c\Sigma_c^T \underbrace{\cancel{U^T U}}_{I}(\Sigma_c^{2^{-1}} U^{-1}) y$$

$$= V_c\Sigma_c U^{-1} y = V_c\Sigma_c U^T y.$$

### 3 Minimum Energy Control

Consider the system

$$\vec{x}(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

Our goal is to reach the target state

$$\vec{x}(5) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

starting at $\vec{x}(0) = 0$.

a) Find the input sequence $u(0), u(1), u(2), u(3), u(4)$ that achieves this with the least possible "energy," as defined by

$$E = u(0)^2 + u(1)^2 + u(2)^2 + u(3)^2 + u(4)^2.$$

Find the value of $E$ for the sequence you computed.

Minimize norm $\|u(0)^2 + u(1)^2 + u(2)^2 + u(3)^2 + u(4)^2\|$.

Write $x(5)$ as a function of $x(4) \dots x(0)$:

$$x(5) - x(0) = C_5 = \begin{bmatrix} b & Ab & A^2b & A^3b & A^4b \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

This satisfies the form $y = Ax$ where we know

$A^T(AA^T)^{-1}y$

This satisfies the form $y = Ax$ which has the

Solution $x^* = A^T(AA^T)^{-1}y$

where $x^* = x(0)$, $y = u$, $A = C$.

b) Even if the wheel and the motor driving it conform perfectly to the model, our inputs still limit the range of velocities. Given that $0 \le u[k] \le 255$, determine the maximum and minimum velocities possible for the wheel. How can you slow the car down?