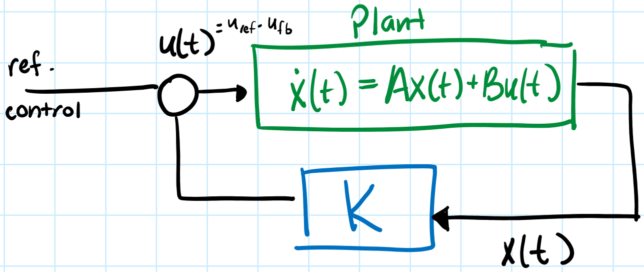
Controllable Canonical Form

Monday, August 3, 2020 1:01 AM

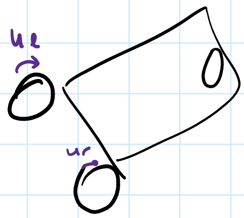


Closed Loop Control

★ Feedback control can be useful even in stable systems to allow faster responses

ex: if $\lambda = -0.01$, can make $\lambda - BK = -1$ to increase rate of change

Car feedback control



We have control of left/right wheel speeds

what if we wanted both wheels to turn at the same rate?

- $u_\ell[k] = \frac{V^* + \beta_\ell}{\theta_\ell} + \frac{k_\ell}{\theta_\ell}\left(d_\ell[k] - d_r[k]\right)$

for $d_\ell[k+1] = d_\ell[k] + \theta_\ell u_\ell[k] - \beta_\ell$.

In matrix form:

$$\begin{bmatrix} d_\ell[k+1] \\ d_r[k+1] \end{bmatrix} = \begin{bmatrix} d_\ell[k] \\ d_r[k] \end{bmatrix} + \begin{bmatrix} V^* \\ V^* \end{bmatrix} + \begin{bmatrix} k_\ell\left(d_\ell[k] - d_r[k]\right) \\ k_r\left(d_\ell[k] - d_r[k]\right) \end{bmatrix}$$

Choose k_ℓ, k_r such that

$$\left|1 + (k_r - k_\ell)\right| < 1 \quad \text{(stabilize)}.$$

Controllable Canonical Form

A special matrix form for $(A - BK)$ in $x_{t+1} = Ax_t + bu_t$

Properties

1. any controllable matrix A can be transformed to CCF
2. the eigenvalues for a matrix in CCF can be easily chosen

1. any controllable matrix A can be transformed to CCF
2. the eigenvalues for a matrix in CCF can be easily chosen

Form

$$x = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \cdots & & 1 & \cdots & 0 \\ \cdots & & & 0 & 1 \\ a_1 & a_2 & \cdots & & a_n \end{bmatrix} x_t + \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} u_t$$

$$\lambda^n - a_n \lambda^{n-1} - a_{n-1}\lambda^{n-2} \cdots\cdots - a_1$$

Characteristic Polynomial

Proof: if matrices A, B define a controllable system, $\exists$ a transformation T such that

$TAT^{-1} = A_c$
$TB = B_c$ A_c B_c are in CCF.

· Define $z_{t+1} = A_c z_t + B_c u_t$ where
$u = -K_c z = -K_c T x$.

Know A has the same eigenvalues as TAT^{-1} (similarities):

↳ $Av = \lambda v$

$(TAT^{-1})(Tv) = TAv = \lambda Tv$

$$T(A - BK)T^{-1} = TAT^{-1} - TBKT^{-1}$$
$$= A_c - B_c K_c \quad \text{so}$$

$A_c - B_c K_c$ has the same eigenvalues as $A - BK$.

Proof: existence of T

If A, B controllable, then $C = [B \ AB \ \cdots \ A^{n-1}B]$ is full rank and invertible.

So C^{-1} exists.

Let q^T = last row of C^{-1} such that

$q^T C = [0 \ 0 \ \cdots \ 1]$

$$q^T C = [0\ 0\ \dots\ 1]$$

then $T = \begin{bmatrix} q^T \\ q^T A \\ \vdots \\ q^T A^{n-1} \end{bmatrix}$ So T exists. $\rightarrow TB = (q^T C)^T$

Controllability Matrix and Change of Basis

For CCF, $\tilde{C} = [\tilde{B}\ \ \tilde{A}\tilde{B}\ \dots\ \tilde{A}^{n-1}\tilde{B}]$

$$= [TB\ \ TA\cancel{T^{-1}T}B\ \dots\ (TA\cancel{T^{-1}})(\cancel{T}AT^{-1})\dots\cancel{T}B]$$

$$\tilde{C} = T[B\ \ AB\ \dots\ A^{n-1}B]$$

$$\tilde{C} = TC$$

$$T = \tilde{C}C^{-1}$$

Feedback in CCF

$$u[t] = \tilde{K}z[t] = \tilde{K}Tx[t] = Kx[t]$$

2 Eiegenvalue Placement in CCF

Consider the following continuous-time system

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

a) Is this system controllable?

Yes- any system in CCF is stable

b) Is the linear continuous time system stable?

$$\lambda^3 + 4\lambda^2 + 3\lambda + 0 = 0$$

$$\lambda(\lambda+3)(\lambda+1) = 0$$

$$\lambda = 0, \quad \lambda = -3, \quad \lambda = -1$$

↑ $Re\{\lambda\} = 0$ so unstable

c) Using state feedback $u(t) = -K\vec{x}(t) = \begin{bmatrix} -k_0 & -k_1 & -k_2 \end{bmatrix} \vec{x}(t)$ place the eigenvalues at $-1, -1, -2$.

Desired system:

$$(\lambda+1)(\lambda+1)(\lambda+2) = 0$$

$$\lambda^3 + 4\lambda^2 + 5\lambda + 2 = 0$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_0 & -k_1-3 & -k_2-4 \end{bmatrix}$$

$$\lambda^3 + (k_2+4)\lambda^2 + (k_1+3)\lambda + k_0 = 0$$

Match coefficients:

$$k_2 + 4 = 4, \quad k_1 + 3 = 5, \quad k_0 = 2$$

$$k_2 = 0, \quad k_1 = 2, \quad k_0 = 2$$

$$\vec{x}[t+1] = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u[t]$$

a) Is this system controllable?

$$C = \begin{bmatrix} b & Ab & A^2 b \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

yes - upper triangular

c) Bring the system to the controllable canonical form

$$\vec{z}[t+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{bmatrix} \vec{z}[t] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[t]$$

using transformation $\vec{z}[t] = T\vec{x}[t]$

$$T = \tilde{C} C^{-1}$$

$$C = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \text{ so } C^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$\tilde{C}$: know $\lambda^3 + 4\lambda^2 + 3\lambda = 0$

so $a_0 = 0,\ a_1 = -3,\ a_2 = -4$

$$\tilde{C} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix}$$

$$T = \tilde{C} C^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix}$$

d) Using state feedback $u[t] = \tilde{K}\vec{z}[t] = \begin{bmatrix} \tilde{k}_0 & \tilde{k}_1 & \tilde{k}_2 \end{bmatrix} \vec{z}[t]$ place the eigenvalues at $0, 1/2, -1/2$.

$$\tilde{A} + \tilde{B}\tilde{K} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \tilde{k}_0 & \tilde{k}_1 - 3 & \tilde{k}_2 - 4 \end{bmatrix}$$

$$\lambda^3 + (4 - k_2)\lambda^2 + (3 - \tilde{k}_1)\lambda - \tilde{k}_0 = \lambda\left(\lambda - \tfrac{1}{2}\right)\left(\lambda + \tfrac{1}{2}\right)$$

$$= \lambda^3 - \tfrac{1}{4}\lambda$$

$$4 - \tilde{k}_2 = 0, \quad 3 - \tilde{k}_1 = -\tfrac{1}{4}, \quad \tilde{k}_0 = 0$$

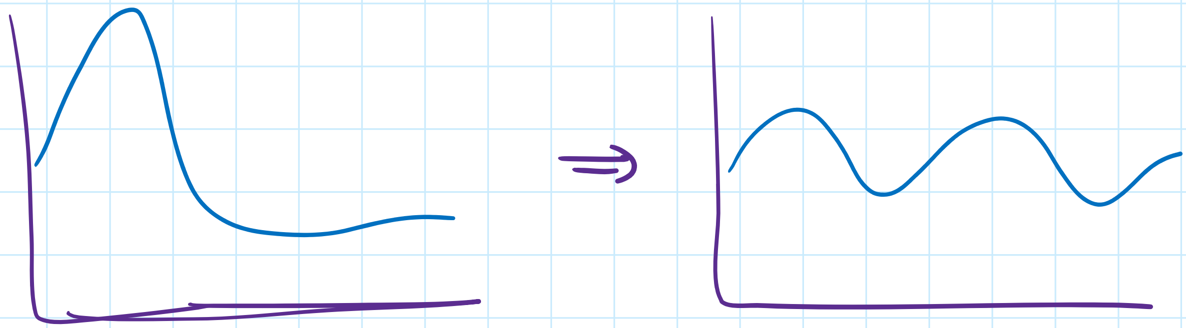
$$\tilde{k}_0 = 0, \quad \tilde{k}_1 = \tfrac{13}{4}, \quad \tilde{k}_2 = 4$$

$$K = \tilde{K}\,T = \begin{bmatrix} 4 \\ -\frac{35}{4} \\ \frac{19}{4} \end{bmatrix} \begin{array}{l} k_0 \\ k_1 \\ k_2 \end{array}$$

Tracking Control

Monday, August 3, 2020 11:35 AM

- Using least squares and control to make a linear system approximate a reference trajectory



Formal Statement:

$$\min_{u_0, x_1, u_1, x_2 \dots} \sum_{t=1}^{n} \| x_t - \hat{x}_t \|_2^2 + \|u_t\|_2^2 \quad \text{for } x_{t+1} = Ax_t + Bu_t$$

$\hat{x}_t$ is the reference trajectory.

Matrix form

$$\begin{bmatrix} B & -I & & & & \\ & A & B & -I & & \\ & & & A & B & -I \\ & & & & \dots & \end{bmatrix} \begin{bmatrix} u_0 \\ x_1 \\ u_1 \\ x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} -Ax_0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

Let $d_k = x_k - \hat{x}_k$:

$$\min_{u_0 \dots} \sum_{t=0}^{n-1} \|u\|_2^2 + \sum_{t=1}^{n} \|d_k\|_2^2$$

$u_0 \dots$ $t=0$ $t=1$

$$\begin{bmatrix} B & -I & & & \\ & A & B & -I & \\ & & \cdots & - & - \end{bmatrix} \begin{bmatrix} u_0 \\ d_1 \\ u_1 \\ d_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \hat{y}_1 - Ax_0 \\ \hat{x}_2 - Ax_1 \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} u_0^* \\ d_1^* \\ u_1^* \\ \vdots \end{bmatrix} = VS^{-1}U^T y$$

or $(A^TA)^{-1}A^Ty$