Discrete Fourier Transform

Tuesday, August 4, 2020 10:22 AM

Roots of Unity

*What complex numbers ω satisfy
$\omega^N = 1$ for $N \in \mathbb{N}$?

Let $\omega = re^{j\theta}$. Then
$\omega^N = r^N e^{jN\theta} = 1.$

Since $e^{jN\theta}$ is on the complex circle, $|e^{jN\theta}| = 1$
so $|r^N| = 1$ as well.
$|r|^N = 1$
$\boxed{r = 1}$

If $r = 1$, $e^{jN\theta} = 1$.
Therefore $N\theta$ can be any multiple of 2π.

$$\underbrace{\theta = 0, \frac{2\pi}{N}, \ldots \left(\frac{2\pi}{N}\right)(N-1)}_{N^{th} \text{ roots of unity}}$$

Properties of Roots of Unity

* Sum : $\sum_{n=0}^{N-1} \omega_N^{n} = \sum_{n=0}^{N-1} \omega_N^{n+1} = 0$

* Conjugate Symmetry: if $\omega^N = 1$, then $\bar{\omega}^N = 1$.

DFT Basis

Let $\{u_0, u_1, \ldots u_{N-1}\}$ be a basis for $\mathbb{C}^N$ where

$$u_k[n] = \frac{1}{\sqrt{N}} \omega_N^{kn}.$$

Example: $\mathbb{C}^3$ DFT basis

• $\omega_N = \omega_3 = e^{\frac{2\pi}{3}j} = -\frac{1}{2} + \frac{\sqrt{3}}{2}j$.

To complete the basis, compute ω_3^0 and ω_3^2:

constant "DC component" → $\omega_3^0 = \left(e^{\frac{2\pi}{3}j}\right)^0 = 1$

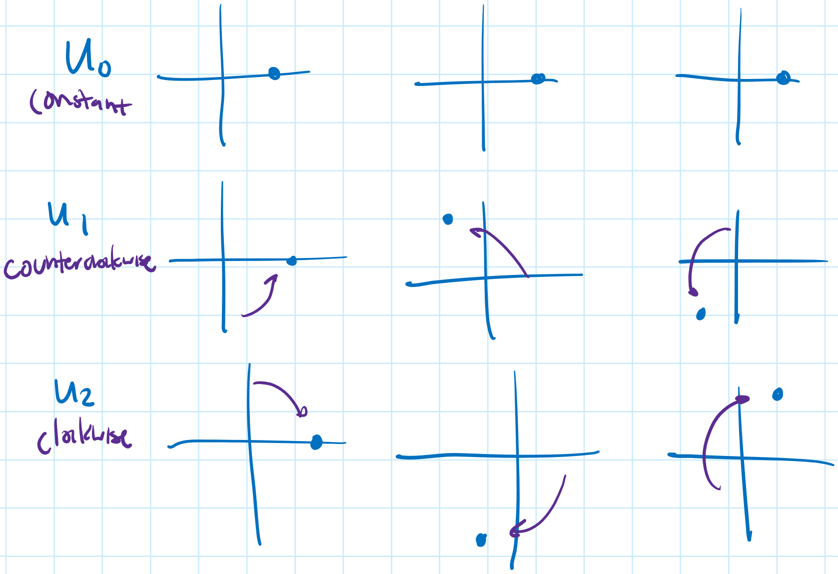
$\omega_3^2 = \left(e^{\frac{2\pi}{3}j}\right)^2 = e^{\frac{4\pi}{3}j} = -\frac{1}{2} - \frac{\sqrt{3}}{2}j$

$$u_0 = \frac{1}{\sqrt{N}}\begin{pmatrix} \omega_N^{0\cdot 0} \\ \omega_N^{0\cdot 1} \\ \omega_N^{0\cdot 2} \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$u_1 = \frac{1}{\sqrt{N}}\begin{bmatrix} \omega_N^{1\cdot 0} \\ \omega_N^{1\cdot 1} \\ \omega_N^{1\cdot 2} \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}j \\ -\frac{1}{2} - \frac{\sqrt{3}}{2}j \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{N}}\begin{bmatrix} \omega_N^{2\cdot 0} \\ \omega_N^{2\cdot 1} \\ \omega_N^{2\cdot 2} \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -\frac{1}{2} - \frac{\sqrt{3}}{2}j \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}j \end{bmatrix}$$

Graphs:



Orthonormality of DFT Basis

Show $\langle u_k, u_{k'} \rangle = 1$ if $k = k'$, 0 otherwise.

- Defintion of inner product:

$$\langle u_k, u_{k'} \rangle = \sum_{n=0}^{N-1} \underbrace{u_k[n]} \overline{u_{k'}[n]}$$

$$= \sum_{n=0}^{N-1} \frac{1}{\sqrt{N}} \omega_N^{kn} \overline{\frac{1}{\sqrt{N}} \omega_N^{k'n}}$$ ← Since $k'n$ is a rotation, $\overline{k'n}$ is a rotation in the opposite direction so $\overline{k'n} = -k'n$.

$$= \sum_{n=0}^{N-1} \frac{1}{\sqrt{N}} \omega_N^{kn} \frac{1}{\sqrt{N}} \omega_N^{-k'n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \omega_N^{(k-k')n}$$

✱ if $k=k'$, this equal $\frac{1}{N}\sum^{N-1} 1 = \frac{N}{N} = 1$

$= \frac{1}{N}\sum_{n=0} \omega_N$

- if $k=k'$, this equals $\frac{1}{N}\sum_{n=0}^{N-1} 1 = \frac{N}{N} = 1$
- if $k \neq k'$, then by the property of DFT basis sums this equals 0.

Let F^* be a matrix whose columns form the DFT basis:

- F^* is unitary $(F^* = F^{-1})$
- $$F^* = \frac{1}{\sqrt{N}}\begin{pmatrix} w_0 & w_0 & \cdots & & w_0 \\ w_0 & w_1 & w^2 & \cdots & w^{N-1} \\ \vdots & \vdots & & \ddots & \\ w_0 & w^{N-1} & & & w^{(N-1)(N-1)} \end{pmatrix} \leftarrow \begin{matrix} F_{nk} = \frac{1}{\sqrt{N}}\omega_N^{-kn} \\ F^*_{nk} = \frac{1}{\sqrt{N}}\omega_N^{kn} \end{matrix}$$

Synthesis equation: $\overset{\text{time}}{x} = F^* X^{\leftarrow \text{freq.}}$

Analysis equation: $\underset{\text{frequency domain}}{X} = F \underset{\text{time domain}}{x}$

DFT Properties

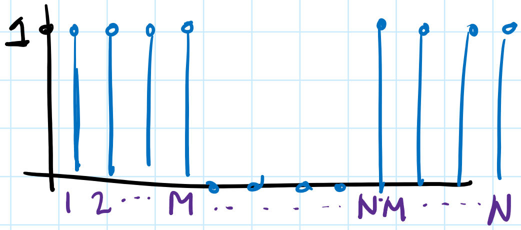
Reminder: $F^*_{nk} = \frac{1}{\sqrt{N}}\omega_N^{nk}$ and $F_{nk} = \frac{1}{\sqrt{N}}\omega_N^{-nk}$.

and $\omega_N = e^{\frac{2\pi}{N}j}$.

1. Linearity: if $a, b \in \mathbb{C}$, then $F(ax+by) = a(Fx) + b(Fy)$
2. Conserves Energy: $\|Fx\|^2 = \|x\|^2$ (Parseval's Theorem)
3. Conjugate Symmetric: if $x = \bar{x}$, then $X[n] = \overline{X[-n]} = \overline{X[N-n]}$

DFT of a Square Wave

- Rectangular Pulse:

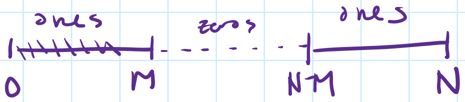


$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{matrix} \}M \\ \}N-2M-1 \\ \}M \end{matrix}$$

Calculate DFT using analysis equation $X = Fx$:

$$X[n] = \underbrace{\sum_{k=-M}^{M} \overline{u}_k}_{} = \frac{1}{\sqrt{N}} \sum_{k=-M}^{M} \omega^{-kn} = \frac{1}{\sqrt{N}} \sum_{k=-M}^{M} (\omega^{n})^{k}$$ using conjugate symmetry

summation:



$= 2M$ length

Geometric sum formula:

$$= \frac{1}{\sqrt{N}} \left(\frac{(\omega^n)^{-M} - (\omega^n)^{M} (\omega^n)^{1}}{1 - (\omega)^n} \right)$$

$$= \frac{1}{\sqrt{N}} \left(\frac{\omega^{n \cdot -M} - \omega^{n \, M+1}}{\omega^{n \cdot 0} \quad \omega^{n}} \right) \left(\frac{\omega^{n \cdot -\frac{1}{2}}}{\omega^{n \cdot -\frac{1}{2}}} \right)$$

$$= \frac{1}{\sqrt{N}} \left(\frac{\omega^{n \cdot -M - \frac{1}{2}} - \omega^{n \, M + \frac{1}{2}}}{\omega^{n - \frac{1}{2}} - \omega^{n \frac{1}{2}}} \right)$$ ← complex conjugate differences

$$= \frac{1}{\sqrt{N}} \left(\frac{e^{-\frac{2\pi n j}{N}(2M+1)} - e^{\frac{2\pi n j}{N}(2M+1)}}{e^{-\frac{\pi n j}{N}} - e^{\frac{\pi n j}{N}}} \right)$$ ← sub $\omega = e^{\frac{-2\pi j}{N}}$

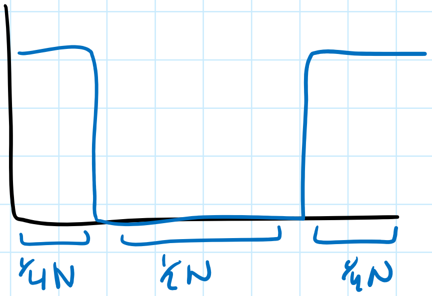
$$= \frac{1}{\sqrt{N}} \left(\frac{\cancel{-2j} \sin\left(\frac{\pi n}{N}(2M+1)\right)}{\cancel{-2j} \sin\left(\frac{\pi n}{N}\right)} \right)$$ ← sub $\sin x = \frac{1}{2j}(e^{jx} - e^{-jx})$

Since $\frac{\pi n}{N} \approx 0$, $\sin \frac{\pi n}{N} \approx \frac{\pi n}{n}$

$$\approx \frac{N}{\sqrt{N}} \, \frac{\sin\left(\frac{\pi n}{N}(2M+1)\right)}{\pi n} \cdot \frac{2M+1}{2M+1}$$

$$\approx \frac{2M+1}{\sqrt{N}} \operatorname{sinc}\left(\frac{2M+1}{N} n\right)$$ ← $\operatorname{sinc} x = \lim_{t \to x} \frac{\sin \pi t}{\pi t}$

If x is a square wave:



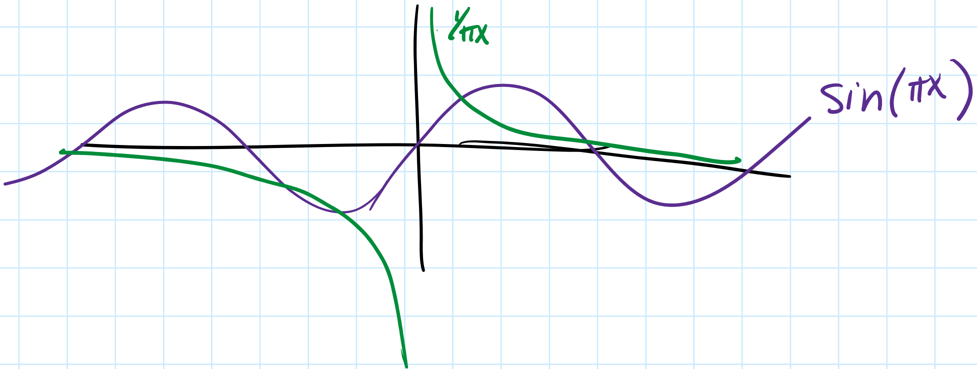
$M = N/4$

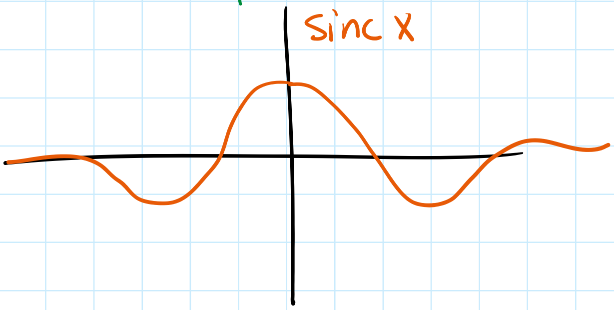
$$X[n] \approx \frac{2\left(\frac{N}{4}\right)+1}{\sqrt{N}} \operatorname{sinc}\left(\frac{2\left(\frac{N}{4}+1\right)}{N} n \right)$$

$$\approx \frac{\sqrt{N}}{2}\text{sinc}\left(\frac{1}{2}n\right) \leftarrow \text{if } N \text{ is large, } 1 \text{ is insignificant}$$

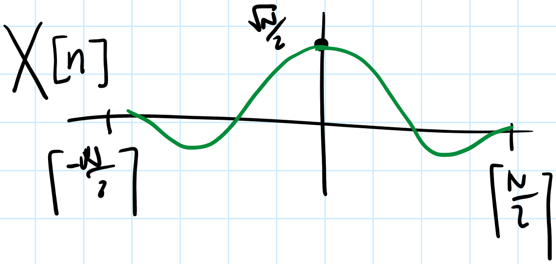
Properties of sinc(x)

$$\text{sinc}(x) = \lim_{t \to x} \frac{\sin(\pi t)}{\pi t}$$





Graph of DFT of Square wave



The DFT of a square is a sinc and the DFT of a sinc is a square.

1 Roots of Unity

The DFT is a coordinate transformation to a basis made up of roots of unity. In this problem we explore some properties of the roots of unity. An Nth root of unity is a complex number z satisfying the equation $z^N = 1$ (or equivalently $z^N - 1 = 0$).

a) Show that $z^N - 1$ factors as

$$z^N - 1 = (z-1)(\sum_{k=0}^{N-1} z^k).$$

$$z^N - 1 = z\sum_{k=0}^{N-1} z^k - \sum_{k=0}^{N-1} z^k$$

$$= \sum_{k=0}^{N-1} z^{k+1} - \sum_{k=0}^{N-1} z^k$$

$$= (\cancel{z} + \cancel{z^2} \ldots + z^N) - (1 + \cancel{z} + \cancel{z^2} \ldots + \cancel{z^{N-1}})$$

$$= z^N - 1$$

b) Show that any complex number of the form $\omega_k = e^{j\frac{2\pi}{N}k}$ for $k \in \mathbb{Z}$ is an N-th root of unity.

Euler's Formula: $\omega_k^N = e^{j2\pi k} = \cos(2\pi k) + j\sin(2\pi k)$

$$= 1 + 0$$

$$= 1$$

2 DFT of pure sinusoids

a) Consider the continuous-time signal $x(t) = \cos\left(\frac{2\pi}{3}t\right)$. Suppose that we sampled it every 1 second to get (for $n = 3$ time steps):

$$x[n] = \begin{bmatrix} \cos\left(\frac{2\pi}{3}(0)\right) & \cos\left(\frac{2\pi}{3}(1)\right) & \cos\left(\frac{2\pi}{3}(2)\right) \end{bmatrix}^T.$$

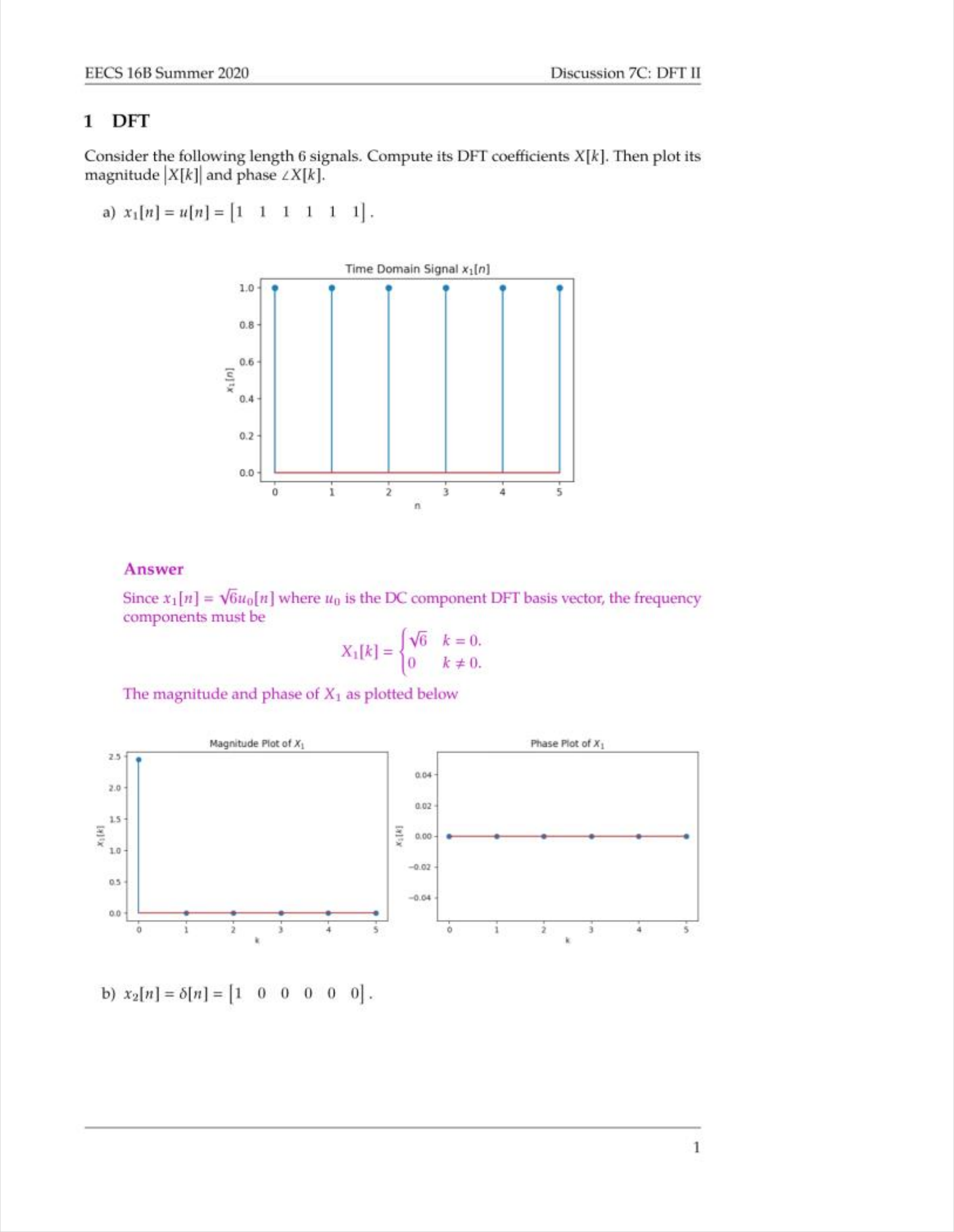
Compute $\vec{X}[k]$ and the basis vectors $\vec{u}_k$ for this signal.

$$F_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-j\frac{2\pi}{3}} & e^{-j\frac{4\pi}{3}} \\ 1 & e^{j\frac{4\pi}{3}} & e^{-j\frac{8\pi}{3}} \end{bmatrix} \qquad F^* = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{j\frac{2\pi}{3}} & e^{j\frac{4\pi}{3}} \\ 1 & e^{j\frac{4\pi}{3}} & e^{j\frac{8\pi}{3}} \end{bmatrix}$$

$$\vec{X} = F\vec{x} = F\begin{bmatrix} 1 \\ \frac{1}{2}(e^{j\frac{2\pi}{3}} + e^{-j\frac{2\pi}{3}}) \\ \frac{1}{2}(e^{j\frac{4\pi}{3}} + e^{-j\frac{4\pi}{3}}) \end{bmatrix}$$

$$= \frac{\sqrt{3}}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

ans7C



1 DFT

Consider the following length 6 signals. Compute its DFT coefficients $X[k]$. Then plot its magnitude $|X[k]|$ and phase $\angle X[k]$.

a) $x_1[n] = u[n] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

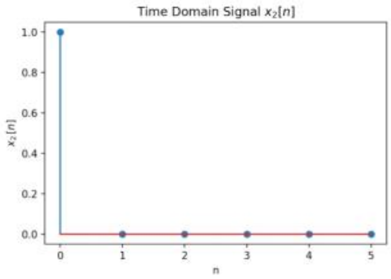
Answer

Since $x_1[n] = \sqrt{6}u_0[n]$ where u_0 is the DC component DFT basis vector, the frequency components must be

$$X_1[k] = \begin{cases} \sqrt{6} & k = 0. \\ 0 & k \neq 0. \end{cases}$$

The magnitude and phase of X_1 as plotted below

b) $x_2[n] = \delta[n] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

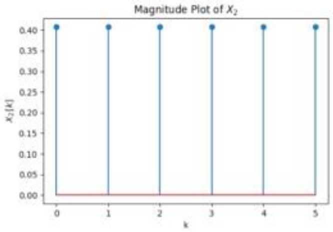


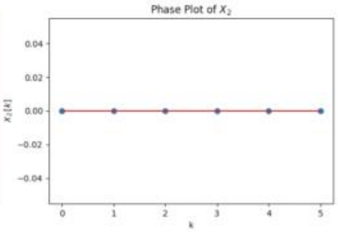
Answer

We can compute the frequency components by multiplying by the matrix $F = U^*$. Since $\delta[n]$ is zero for $n > 0$, the frequency components will be the first column of F.

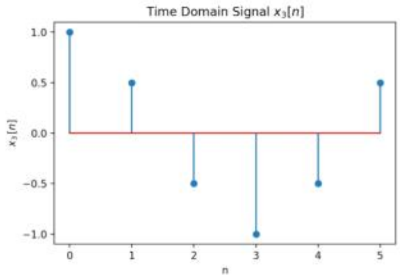
$$X_2[k] = Fx_2[n] = \frac{1}{\sqrt{6}}\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}$$

The magnitude and phase of X_2 as plotted below





c) $x_3[n] = \cos\left(\frac{2\pi}{6}n\right)$ for $n = 0, 1, \ldots, 5$.



Answer

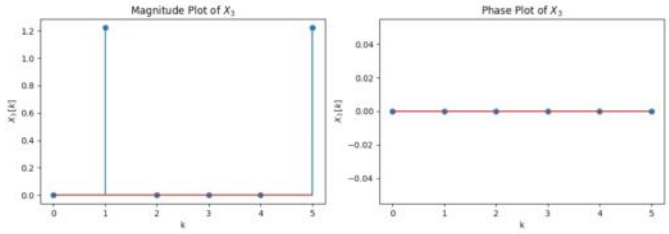
$$\cos\left(\frac{2\pi}{6}n\right) = \frac{1}{2}e^{j\frac{2\pi}{6}n} + \frac{1}{2}e^{-j\frac{2\pi}{6}n}$$

$$u_k[n] = \frac{1}{\sqrt{6}}e^{j\frac{2\pi}{6}kn}$$

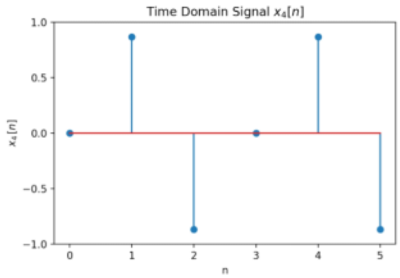
$$\vec{x}_3 = \frac{\sqrt{6}}{2}(\vec{u_1} + \vec{u_5})$$

$$X_3[k] = \begin{cases} \frac{\sqrt{6}}{2} & k = 1, 5. \\ 0 & k \neq 1, 5. \end{cases}$$

The magnitude and phase of X_3 as plotted below



d) $x_4[n] = \sin\left(\frac{4\pi}{6}n\right)$ for $n = 0, 1, \ldots, 5$.

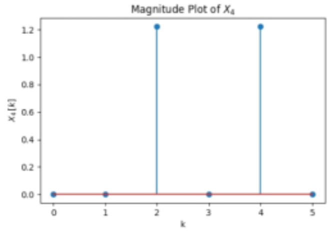


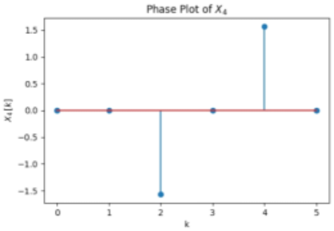
Answer

$$\sin\left(\frac{4\pi}{6}n\right) = \frac{1}{2j}e^{j\frac{4\pi}{6}n} - \frac{1}{2j}e^{-j\frac{4\pi}{6}n}$$
$$u_k[n] = \frac{1}{\sqrt{6}}e^{j\frac{2\pi}{6}kn}$$
$$\vec{x}_4 = \frac{\sqrt{6}}{2j}(\vec{u}_2 - \vec{u}_4)$$

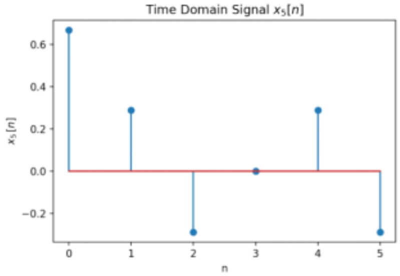
$$X_4[k] = \begin{cases} \frac{-\sqrt{6}j}{2} & k = 2 \\ \frac{\sqrt{6}j}{2} & k = 4 \\ 0 & k \neq 2, 4. \end{cases}$$

The magnitude and phase of X_4 as plotted below





e) $x_5[n] = \frac{2}{3}x_2[n] + \frac{1}{3}x_4[n]$

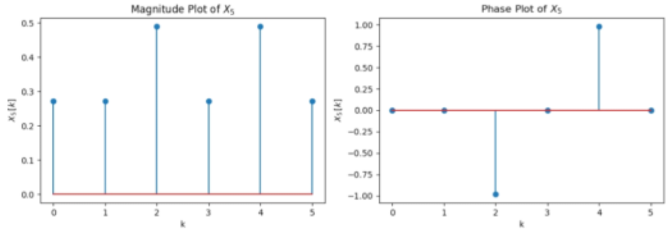


Answer

Since the DFT is linear, $X_5[k] = \frac{2}{3}X_2[k] + \frac{1}{3}x_4X_4[k]$. Therefore, we can write out the DFT coefficients of x_5 as

$$\begin{aligned} X_5[k] &= \frac{2}{3}\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 & 0 & \frac{-\sqrt{6}j}{2} & 0 & \frac{\sqrt{6}j}{2} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3\sqrt{6}} & \frac{2}{3\sqrt{6}} & \frac{2}{3\sqrt{6}} - \frac{\sqrt{6}j}{6} & \frac{2}{3\sqrt{6}} & \frac{2}{3\sqrt{6}} + \frac{\sqrt{6}j}{6} & \frac{2}{3\sqrt{6}} \end{bmatrix} \end{aligned}$$

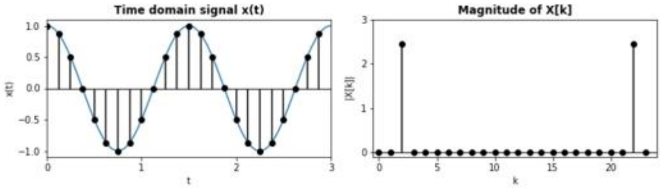
The magnitude and phase of X_5 as plotted below



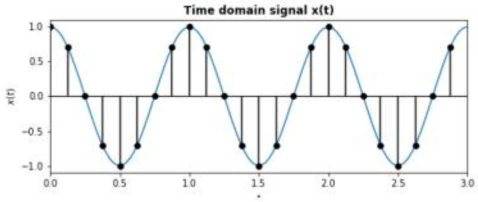
2 DFT Sampling Matching

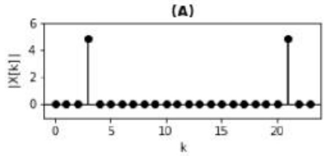
Select the correct answer from the multiple choice options provided and give some justification.

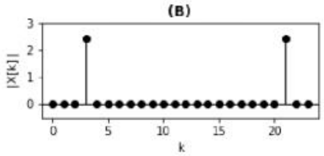
a) A sampled time domain signal and its DFT coefficients are given below:

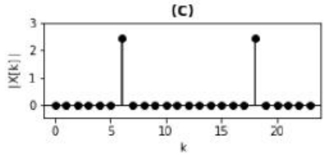


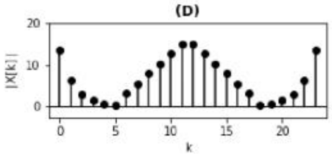
Now given the following time domain signal, which of the options below shows the correct DFT coefficient magnitudes?







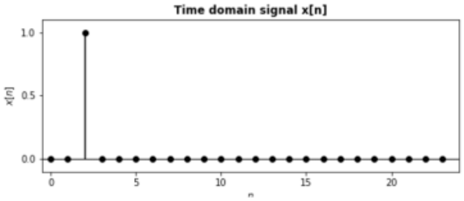




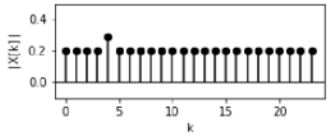
Answer

The correct DFT coefficients are shown in B. The new signal completes 3 full cycles during the discrete sequence. Its amplitude is the same as the first sequence.

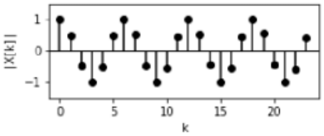
b) Given the time domain signal below, which of the options below shows the correct DFT coefficient magnitudes?



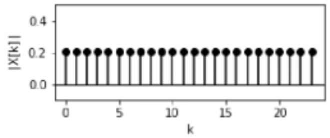
(A)



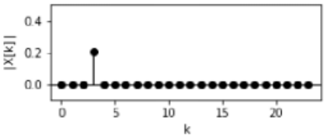
(B)



(C)



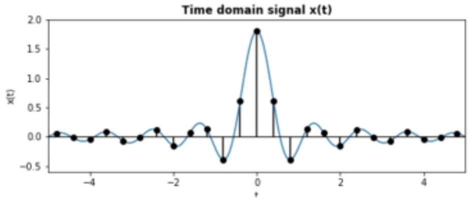
(D)

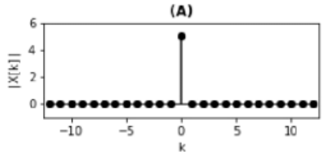


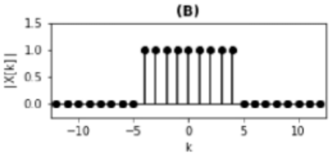
Answer

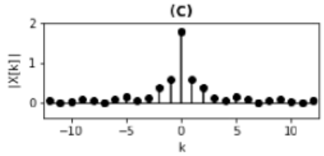
The correct DFT coefficients are shown in C. The DFT coefficients of a unit impulse are all $\frac{1}{\sqrt{N}}$. This impulse has been shifted, so the DFT coefficients have varying phase and are not purely real; however, their magnitudes are still uniformly $\frac{1}{\sqrt{N}}$.

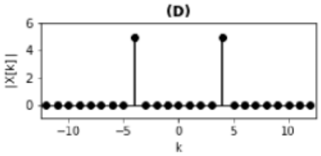
c) Given the time domain signal below, which of the options below shows the correct DFT coefficient magnitudes?







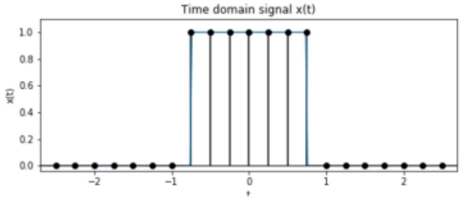




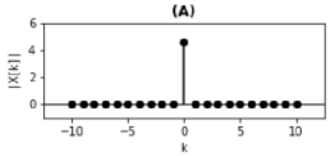
Answer

The correct DFT coefficients are shown in B. We saw in lecture that the DFT of a sinc is a boxcar and vice versa.

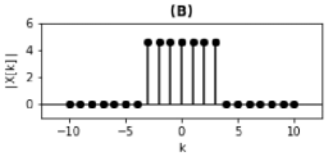
d) Given the time domain signal below, which of the options below shows the correct DFT coefficient magnitudes?



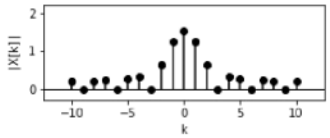
(A)



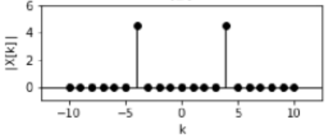
(B)



(C)



(D)



Answer

The correct DFT coefficients are shown in C. We saw in lecture that the DFT of a sinc is a boxcar and vice versa.