Discrete-Time Systems

- uses difference equations instead of differential eq's

$$\Downarrow$$

$$x(t+1) = x_{t+1} = f(x(t), u(t)) \text{ for } t \in \{0,1,2\ldots\}$$

- equilibrium points:

$$X_{eq}(u) = \{ x \mid f(x,u) = x \}$$

If linear,

$$X_{eq}(u) = \{ x \mid (I-A)x = Bu \}$$

Example:

Let $p(t)$ = # professors, $r(t)$ = industry researchers

γ = fraction of PhD's who become profs

α = fraction of field that leave

$$\Downarrow$$

# of PhD's per prof

$$p(t+1) = (1-\alpha)p(t) + \gamma p(t) u(t)$$

$$r(t+1) = (1-\alpha)r(t) + (1-\gamma)p(t)u(t)$$

Let state vector $x(t) = \begin{bmatrix} p(t) \\ r(t) \end{bmatrix}$.

make $p(t+1) = p(t)$, $r(t+1) = r(t)$

If $u(t) = 0$, $x(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (no PhD's).

If $u(t) \neq 0$, we need to find $X_{eq}(u) = \left\{ p, r \;\middle|\; \begin{matrix} (1-\alpha+\gamma u)p - p = 0 \\ -\alpha r + (1-\gamma)pu = 0 \end{matrix} \right\}$

Linearization!

If x^*, u^* is equilibrium:

$$x(t+1) \approx x^* + \nabla_x f(x^*, u^*)\, \delta x(t) + \nabla_u f(x^*, u^*)\, \delta u(t)$$

2 Dirty Dishes

I am a trip planner who lodges travellers at Bob's Bed and Breakfast. At the beginning of each day, Bob will do half of the dirty dishes in the sink. During the day, each of his guests will use 4 pounds of dishes minus an eighth pound of dishes for each pound of dishes already in the sink at the beginning of the day (as Bob's kitchen gets too messy).

a) What is the state vector for Bob's kitchen sink system? What are the inputs? Write out the state space model.

- $x[n+1] = \frac{1}{2}x[n] + \left(4 - \frac{1}{8}x[n]\right)u[n]$

number of dirty dishes

previous day

# of guests

*Not a linear system because of the cross-term $-\frac{1}{8}x[n]u[n]$

c) On Wednesday morning (before Bob gets up), there are 4 pounds of dishes in the sink. On Wednesday, Bob has 4 guests, and on Thursday, he has 5 guests. How many pounds of dishes are in the sink after Thursday?

Let t be Weds.

$$\begin{cases} x[t+1] = \frac{1}{2}x[t] + \left(4 - \frac{1}{8}x[t]\right)u[t] \\ x[t+2] = \frac{1}{2}x[t+2] + \left(4 - \frac{1}{8}x[t+1]\right)u[t+1] \end{cases}$$

$x[t] = 4$, $u[t] = 4$, $u[t+1] = 5$. Plug in to get

$$\begin{cases} x[t+1] = \frac{1}{2}(4) + \left(4 - \frac{1}{8}(4)\right)4 = 16 \\ x[t+2] = \frac{1}{2}(16) + \left(4 - \frac{1}{8}\cdot 16\right)\cdot 5 = 18 \end{cases}$$

e) Now suppose 5 guests come to Bob's kitchen every day. At the equilibrium state, how many pounds of dishes will remain in the sink?

x^* is an equilibrium point if $x[t] = x[t+1] = \ldots = x^*$

$\hookrightarrow \dot{x}(t) = 0$ in continuous systems

"State not changing"

Plug in x^* to get

"State not changing"

Plug in x^* to get

$$x[t+1] = x^* = \tfrac{1}{2}x^* + \left(4 - \tfrac{1}{8}x^*\right)u[t]$$

$$\tfrac{1}{2}x^* = 4u[t] - \tfrac{1}{8}x^* u[t]$$

$$x^*\left(\tfrac{1}{2} + \tfrac{1}{8}u[t]\right) = 4u[t]$$

$$x^* = \frac{4u[t]}{\tfrac{1}{2} + \tfrac{1}{8}u[t]}$$

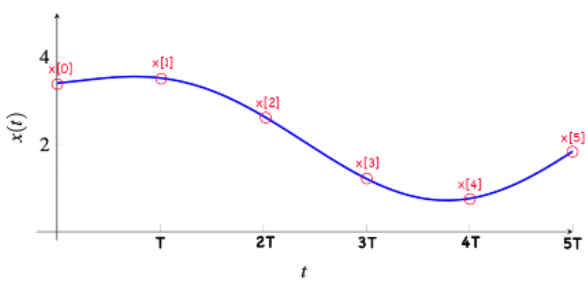
3 Differential equations with piecewise constant inputs

Let $x(\cdot)$ be a solution to the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda\left(x(t) - u(t)\right). \tag{4}$$

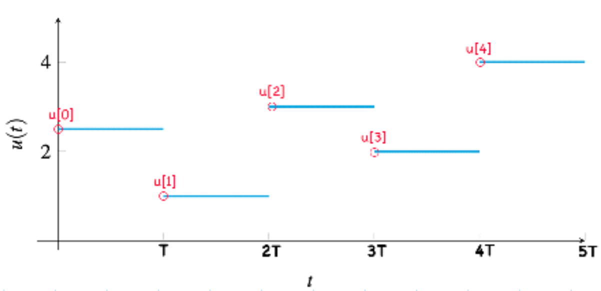
Let $T > 0$. Let $x[\cdot]$ "sample" $x(\cdot)$ as follows:

$$x[n] = x(nT). \tag{5}$$



Assume that $u(\cdot)$ is constant between samples of $x(\cdot)$, i.e.

$$u(t) = u[n] \quad \text{when} \quad nT \le t < (n+1)T. \tag{6}$$



a) We will approach solving this differential equation iteratively in intervals of size T. What is the solution $x(t)$ for $t \in [0, T)$? In terms of $x(0)$ and $u[0]$.

b) Using your solution from the previous part, sample it at $t = T$ to write $x[1]$ in terms of $x[0]$ and $u[0]$.

c) Now for a general time-step n, write $x[n+1]$ in terms of $x[n]$ and $u[n]$. Conclude that the sampled system of a continuous-time linear system is in fact a discrete-time linear

sol: a) $\frac{d}{dt}x(t) = \lambda(x(t) - \underbrace{u(t)}_{=u[0],\ \text{for } t\in[0,T]}) = \lambda(x(t) - u[0])$, for each $t \in [0,T]$.

$\Rightarrow$ Define $\bar{x}(t) := x(t) - u[0]$ $\Rightarrow$ $\frac{d}{dt}\bar{x}(t) = \frac{d}{dt}x(t) = \lambda(x(t) - u[0]) = \lambda\bar{x}(t)$

2

$\Rightarrow$ $\bar{x}(t) = e^{\lambda t}\bar{x}(0) = e^{\lambda t}(x(0) - u[0])$

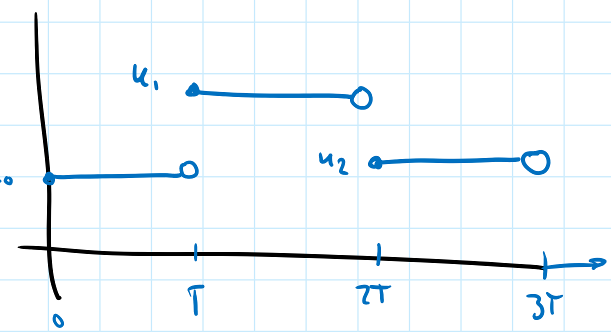
$\Rightarrow$ $x(t) = \bar{x}(t) + u[0] = e^{\lambda t}(x(0) - u[0]) + u[0] = e^{\lambda t}x[0] + (1 - e^{\lambda t})u[0]$.

Digital Control of Continuous Systems

- Control a continuous system using discrete inputs
  - ↳ sample frequency $= \frac{1}{T}$ such that $u(t)$ is chosen at

$$x(0),\ x(T),\ x(2T) \ldots.$$

zero order hold: maintain constant input until next sample



Discretization

Given a linear system

$$\dot{x}(t) = Ax(t) + Bu(t),$$

find an equivalent discrete time system

$$x_{n+1} = A_d x_n + B_d u_n \quad \text{where}$$

$u(t)$ is a zero order hold.

Scalar case:

$$\dot{x}(t) = \lambda x(t) + bu(t)$$

Find solution to $x(kT) = x_k$:

$$x(t) = e^{\lambda(t-kT)} x_k + \int_{kT}^{t} e^{\lambda(t-\tau)} bu(\tau)\, d\tau$$

Since zero-order hold, we can set $u(\tau)$ to a constant u_k.

For $t = kT + T$:

$$x(kT+T) = e^{\lambda(kT-kT+T)} x_k + \int_{kT}^{kT+T} e^{\lambda(kT+T-\tau)} d\tau \cdot bu_k$$

$$= e^{\lambda T} x_k + \int_0^T e^{\lambda(T-\tau)} d\tau \cdot bu_k$$

$$\underline{x(kT+T) = e^{\lambda T} x_k + b\left(\frac{e^{\lambda T}-1}{\lambda}\right) u_k}$$

Vector case:

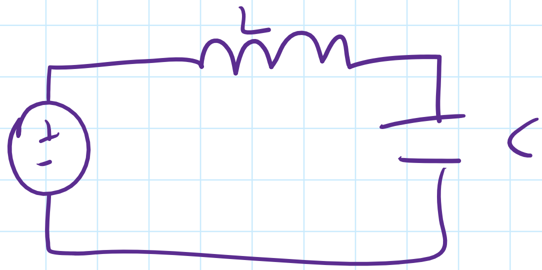
If A is diagonal, we can solve each dimension independently like a scalar case.

If A is not diagonal, diagonalize it!

$$z_{k+1} = T A_d T^{-1} z_k + T B_d u_k.$$

$$x_{k+1} = T z_{k+1}$$

Example: LC Circuit



$$\dot{x}(t) = A\begin{bmatrix} I_L \\ V_C \end{bmatrix} + B\,V_{in}(t)$$

$$I = C\frac{dV}{dt}$$

$$V_L = L\frac{dI}{dt}$$

$$V_{in} - V_L - V_C = 0$$

$$V_{in} - L\frac{dI}{dt} - V_C = 0$$

$$\frac{dI}{dt} = \frac{V_{in}}{L} - \frac{V_C}{L}$$

$$\frac{dV}{dt} = \frac{I}{C}$$

$$\begin{bmatrix} I_L \\ V_C \end{bmatrix}' = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} I_L \\ V_C \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} V_{in}(t)$$

→ get eigenvals and eigenvecs

→ diagonalize like normal

→ get rotation matrix

3 Discretization

Consider a cart of mass M, pushed with a force $u(t)$ with position, $x(t)$, and velocity, $v(t)$. Hence, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = v(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = \frac{u(t)}{M}$$

We will apply a constant input between any time $t \in [nT, nT + T)$. Here T is our time between samples.

a) Find a discretized system of equations for this system.

$\dot{x} = V(t)$ so $x(t) - x(nT) = \int_{nT}^{t} V(\tau)\,d\tau$

$\dot{V} = \frac{u(t)}{M}$ so $V(t) - V(nT) = \int_{nT}^{t} \frac{u(\tau)}{M}\,d\tau$

$U(\tau)$ is constant, so

$$V(t) - V(nT) = \frac{u[nT]}{M}(t - nT).$$

Discretize by replacing $t = (n+1)T$.

$$v[n+1] = v[n] + \frac{u[n]}{M}(T(n+1) - nT)$$
$$= v[n] + \frac{u[n]}{M}T$$

Solve x.

$$x(t) - x(nT) = \int_{nT}^{t} V(nT) + \frac{u(nT)}{M}\underbrace{(\theta - nT)}_{=\tau}\,d\theta$$
$$= u(nT)(t - nT) + \int_{0}^{t-nT}\frac{u(nT)}{M}\tau\,d\tau$$

$$x(t) = x(nT) + v(nT)(t - nT) + \frac{1}{2}\frac{u(nT)}{M}(t - nT)^2$$

$$x(n+1) = x[n] + v[n]T + \frac{1}{2}\frac{u[n]}{M}T^2$$

Matrix form:

$$\begin{bmatrix} x(n+1] \\ v[n+1] \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x[n] \\ v[n] \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\frac{T^2}{M} \\ \frac{T}{M} \end{bmatrix} u[n]$$