Inner Product Spaces

- An inner product on a vector space V is a function of a pair of vectors with the following properties:

i) $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$ (additive in 1st arg)
ii) $\langle \alpha x, y\rangle = \alpha\langle x,y\rangle$ (scaling in 1st arg)
iii) $\overline{\langle x,y\rangle} = \langle y,x\rangle$ (symmetric conjugates)
iv) $\langle x,x\rangle$ is real and non-negative (positive-definitive)

Inner product space: a vector space over $\mathbb{C}$ equipped with inner product

- Conjugate linear in 2nd argument: (proof)

$$\langle x, y+\alpha z\rangle = \overline{\langle y+\alpha z, x\rangle} \quad (iii)$$
$$= \overline{\langle y,x\rangle + \alpha\langle z,x\rangle} \quad (ii, i)$$
$$= \langle x,y\rangle + \bar{\alpha}\langle x,z\rangle \quad (iii)$$

Standard inner product:

★ Define $\langle u,v\rangle_{std} = v^*u = \sum_{i=1}^{n} u_i\overline{v_i}$

Vector length: $\|v\| = \sqrt{\langle v,v\rangle}$

Complex Inner Product

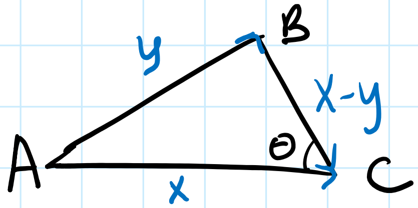
Positive-definite inner product:

$$\langle \vec{u},\vec{v}\rangle = \sum_{n=1}^{n} u_i\overline{v_i} = \overline{v}^T u = v^* u$$

- v^* = complex conjugate transpose
- $\langle u,v\rangle = \langle v,u\rangle$

- V^* = complex conjugate transpose
- $\langle u,v\rangle = \langle v,u\rangle$

Example: Derivation of Law of Cosines



$$\Rightarrow \|x-y\|^2 = \langle x-y, x-y\rangle$$

$$= \langle x, x-y\rangle - \langle y, x-y\rangle \quad \text{(i)}$$

$$= \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle$$

$$= \|x\|^2 + \|y\|^2 - \langle x,y\rangle + \overline{\langle x,y\rangle} \quad \text{(ii)}$$

$$= \|x\|^2 + \|y\|^2 - 2\mathrm{Re}\{\langle x,y\rangle\}\cdot\frac{\|x\|\|y\|}{\|x\|\|y\|} \quad \cos\theta$$

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\theta$$

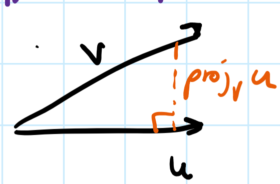
Orthogonality

A set of vectors $\{v_1 \dots v_i\}$ is orthogonal if $\langle v_i, v_j\rangle = 0$ for all distinct i, j $(i \neq j)$.

It is orthonormal if $\langle v_n, v_n\rangle = 1$ for all $1 \le n \le i$

If $\{v_1 \dots v_n\}$ are orthogonal, then $\sum_{i=1}^{n} \|v_i\|^2 = \|\sum_{i=1}^{n} v_i\|^2$

Orthogonal Projection: $\mathrm{proj}_v u = \frac{\langle u,v\rangle}{\langle v,v\rangle} v$



Cauchy-Schwarz Inequality: $|\langle u,v\rangle| \le \|u\|\|v\|$

in $\mathbb{R}^n$, $\langle u,v\rangle = \|u\|\|v\|\cos\theta$ and $\cos\theta \le 1$.

Triangle Inequality: $\|u+v\| \le \|u\| + \|v\|$ if $u, v \in V$.

see law of cosines proof

Unitary matrix U has orthonormal columns, such that

Unitary matrix U has orthonormal columns, such that

$$U^*U = I$$

↳ The eigenvalue of an orthogonal operator $f: V \to V$ is ± 1.
Therefore, unitary matrices are rotation/reflection matrices.

Gram-Schmidt Process

- input: a set of linearly independent vectors
- output: an orthonormal set of vectors with the same span
  - ↳ orthogonal, have a length of 1

Demo: Gram-Schmidt process for 3 vectors $\{V_1, V_2, V_3\}$

- Find unit vector for $\text{span}(u_1) = \text{span}(v_1)$

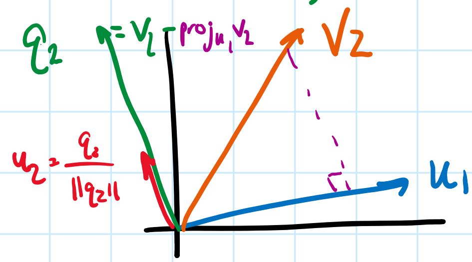
  $u_1 = kv_1$ such that $|kv_1| = 1$

  $= \frac{v_1}{\|v_1\|} = \frac{v_1}{\sqrt{\langle v_1, v_1 \rangle}}$

$\langle u_1, u_2 \rangle = 0$

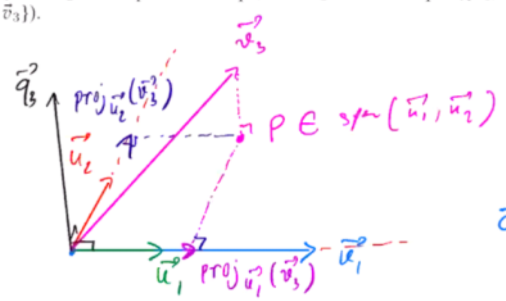
- Find u_2 unit vector where $\text{span}\{u_1, u_2\} = \text{span}\{v_1, v_2\}$ and $u_1 \perp u_2$

  -remove $\text{proj}_{u_1}(V_2)$ from V_2 to get only the orthonormal basis



$$* \ \text{proj}_{u_1} V_2 = \langle u_1, V_2 \rangle V_2$$

c) Now given $\vec{u}_1$ and $\vec{u}_2$ in the previous steps, find $\vec{u}_3$ such that span($\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$) = span($\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$).



$$\vec{q}_3 = \vec{v}_3 - \text{proj}_{\vec{u}_1}(\vec{v}_3) - \text{proj}_{\vec{u}_2}(\vec{v}_3)$$

$$\vec{u}_3 = \frac{\vec{q}_3}{\|\vec{q}_3\|}$$

- Find u_3 using the same process. Generalized:
  - Find $u_1 = \frac{v_1}{\|v_1\|}$
  - For each u_i, $2 \le i \le n$:

$$q_i = v_i - \sum_{j=1}^{i-1} \text{proj}_{\vec{u}_j}(\vec{v}_i)$$
$$= v_i - \sum_{j=1}^{i-1} \langle v_j, u_j \rangle u_j$$

Normalize q_i to get $u_i = \frac{q_i}{\|q_i\|}$

Example: $\{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

- $u_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
- $q_2 = v_2 - \text{proj}_{u_1} v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\rangle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
  $= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
- $u_2 = q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
- $q_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = u_3$

- Gram-Schmidt is dependent on the order: choosing a different v_1 will change the result