LTI and Convolution

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Linear Time Invariance

Take a system whose input and output are periodic and discrete:

$$x \to \boxed{\text{System}} \to y$$

This system is linear if $y = Hx$ for a matrix H.

This system is <u>time invariant</u> if for some shift S,

$$\underline{SHx} = HSx \quad \text{so} \quad SH = HS.$$

where S is a left shift 1 spot in time, such that

$$(Sy)[n] = y[n+1]$$

Intuition: you can shift before or after the transformation to get the same result.

Shifts in the Standard Basis

Let δ_k be an <u>impulse</u> at time k such that

$$\delta_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \delta_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and so on.}$$

So $\delta_i = S^{-i}\delta_0 \quad (S^N = I)$

$$\delta_0 \to \boxed{H} \to \underline{h}$$

impulse response: what a time invariant system does to δ_0.

Splitting x into its components:

$$y = Hx = H\sum_{k=0}^{N-1} x[k]\,\delta_k$$

$$= H\sum_{k=0}^{N-1} x[k]\left(S^{-k}\delta_0\right)$$

If H is LTI, then $y = \sum_{k=0}^{N-1} x[k]\left(HS^{-k}\right)\delta_0$

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$= \sum_{k=0}^{N-1} x[k]S^{-k}\underbrace{(H\delta_0)}_{\text{impulse response}}$

$$y = \sum_{k=0}^{N-1} x[k] S^{-k} h$$

or

$$y[n] = \sum_{k=0}^{N-1} x[k]h[n-k]$$

* Convolution Sum

similar $\approx$> $\int_0^t u(\tau)h(t-\tau)d\tau$

Diagonalizing S

Reminder: S is a left circular shift by 1 place:

$$(Sx)[n] = x[n+1]$$

· find eigenvalues of s'. $Sv = \lambda v$

$S^N v = \lambda S^{N-1} v$ ← mult. both sides by S^{N-1}

$= \lambda \cdot \lambda^{N-1} v$ ← by eig property

$= \lambda^N v$

Since the system is circular, $S^N v = v$

$v = \lambda^N v$

$$(\lambda^N - 1)v = 0$$

$\Uparrow$

$\lambda^N = 1$ so $\lambda = \underbrace{\omega^m}_{\text{any root of unity}}, m \in \mathbb{Z}$

$\Downarrow$

Using this,

$$\omega^m v[n] = v[n+1]$$

$v = \underbrace{(\omega^m, \omega^{m+1} \ldots \omega^{m+N-1})}$

↪ DFT basis if normalized by $\frac{1}{\sqrt{N}}$

S and LTI

* If $y = Hx$ is LTI, then $H = p(S)$ for some polynomial p

$$\left(p(S) = \sum_{k=0}^{N} p_k S^k\right)$$

$$H = \begin{bmatrix} h[0] & h[N-1] & \cdots & h[1] \\ h[1] & h[0] & & \vdots \\ \vdots & \vdots & & \vdots \\ h[N-1] & \vdots & & \end{bmatrix}$$

Decompose H by its diagonal stripes.

$$H = h[0]S^0 + h[1]S^{-1} + h[2]S^{-2} + \dots + h[N-1]S$$

$$H = \sum_{k=0}^{N-1} h[k]S^{-k} = \sum_{k=0}^{N-1} h[k]S^{N-k} = p(S)$$

Conclusion: H has the same eigenvectors as S.
H has eigenvalues $p(\lambda)$, where λ is an eigenvalue of S.

$$H u_k = p(\lambda_k) u_k = p(\omega_n{}^k) u_k$$

$$p(\omega^k) = \sum_{\ell=0}^{N-1} h[\ell](\omega^k)^{N-\ell} \quad \overline{\omega^{k\ell}}$$

The eigenvalues of H are $\sqrt{N}\, g[k]$ where $g = Fh$ (impulse response).

☆ if y is the convolution of x ($y = Hx$), then the frequency components of y (Fy) are equal to the frequency response of the impulse response multiplied by the frequency components of x.

$$Fy = (\sqrt{N}\, Fh) \odot Fx$$

$$S = F^* \Omega F \quad \text{where} \quad \Omega = \begin{bmatrix} \omega_0 & & & \\ & \omega_1 & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{bmatrix}$$

3 Is it LTI?

Determine if the following systems are LTI:

a) $y[n] = 4x[n]$

b) $y[n] = 2x[n] - 4$

c) $y[n] = 2x[-2+3n] + 2x[2+3n]$

d) $y[n] = 4^{x[n]}$

e) $y[n] - y[n-1] = x[n]$

f) $y[n] = x[n] + nx[n-1]$

*Check 2 conditions:

1) linearity → let $x = \alpha x_1 + \beta x_2$. Show $H_0(\alpha x_1 + \beta x_2) = H_0 \alpha x_1 + H_0 \beta x_2$

2) time independence → show $(H_0 x[n-n_d])[n] = (H_0 x[n])[n-n_d]$

Let $y = H_0 x$.

ab) show $H_0(\alpha x_1 + \beta x_2)[n] = a(\alpha x_1 + \beta x_2)[n] + b, \quad a,b \in \mathbb{R}$

$$H_0 \alpha x_1[n] + H_0 \beta x_2[n] = a\alpha x_1[n] + a\beta x_2[n] + b$$

$$\alpha(H_0 x_1[n]) + \beta(H_0 x_2[n]) = \alpha(a x_1[n] + b) + \beta(a x_2[n] + b)$$

$$= a\alpha x_1[n] + a\beta x_2[n] + (\alpha_1 + \alpha_2)\beta$$

Therefore, H is linear only if $b = 0$.

a is linear, b is not

Time invariance:

verify $(H_0 \tilde{x})[n] = (H_0 x)[n - n_d]$ for $\tilde{x} = x[n - n_d]$

c) is linear.

Time invariance:

$$H_0 \tilde{x}[n] = 2\tilde{x}[-2+3n] + 2\tilde{x}[2+3n] = 2x[-2+3n-n_d] + 2x[n-n_d]$$

$$H_0 x[n-n_d] \neq 2x[-2+3(n-n_d)] + 2x[2+3(n-n_d))$$

↳ these quantities are not equal, so H is not time invariant.

d) linearity:

$$H_0(\alpha x_1 + \beta x_2) = 4^{\alpha x_1 + \beta x_2} = 4^{\alpha x_1} 4^{\beta x_2} = H_0(\alpha x_1)\, H_0(\alpha x_2)$$

Not linear

f) time invariance:

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$$H_0 \tilde{x}[n] = x[n-n_d] + n\,x[n-n_d-1]$$

$$(H_0 x)[n-n_d] = x[n-n_d] + (n-n_d)\,x[n-n_d-1]$$

Convolution

Given a LTI system $y = Hx$:

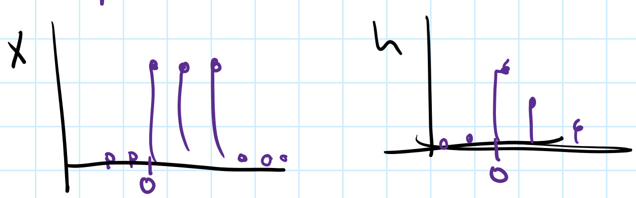
The convolution is

$$(x * h)[i] = \sum_{k=-\infty}^{\infty} x[i]\,h[i-k]$$

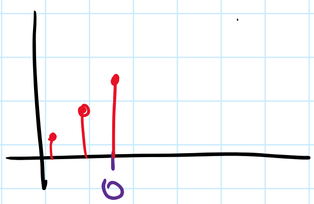
1st signal: keep the same

impulse response: flip at 0 (1st nonzero entry) index, shift by i, and multiply element-wise with x.

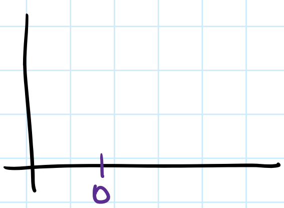
example:



h flipped



$$x[0] \sum_{-\infty}^{\infty} h[-k]$$



Circulant Matrix

- Columns are shifted versions of the 1st column

$$C_h = \begin{bmatrix} h_0 & h_{N-1} & \cdots & h_2 & h_1 \\ h_1 & h_0 & h_{N-1} & & h_2 \\ \vdots & h_1 & h_0 & \ddots & \vdots \\ h_{N-2} & \vdots & \ddots & \ddots & h_{N-1} \\ h_{N-1} & h_{N-2} & \cdots & h_1 & h_0 \end{bmatrix}$$

- For a LTI system $y = Hx$, H is a circulant matrix.

$$\begin{bmatrix} h_{N-1} & h_{N-2} & \cdots & h_1 & h_0 \end{bmatrix}$$

· For a LTI system $y = Hx$, H is a circulant matrix.

Convolution of circulant matrix:

$$\sum_{k=0}^{N-1} \underbrace{h[n-k]}_{= h[n] \text{ shifted down}} x_k$$

$$\cdot\ H = \underbrace{p(S)}_{\text{polynomial representing}} = \sum_{k=0}^{N-1} h[k]\, S^{N-k}$$

polynomial representing left shift: $S = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & & \vdots \\ \vdots & & \ddots & \ddots & & \\ & & & & 1 & 0 \end{bmatrix}$ w/ eigenvalues of N roots of unity, eigenvectors form DFT basis $\rightarrow p(\omega_N^k) = \sqrt{N} F h$

Decomposition of H:

$$H = UDU^{-1} = F^* D F$$

$$y = Hx = F^* D F x$$

$$= F^* p(\omega_N^k) \odot (Fx)$$

$$= F^* (\sqrt{N} F h) \odot \underbrace{Fx}_{= X}$$

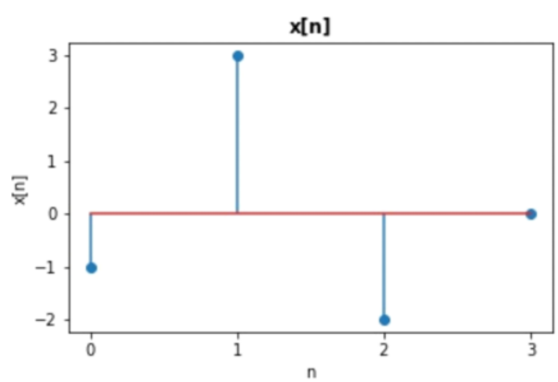
elementwise multiplication $\odot$:

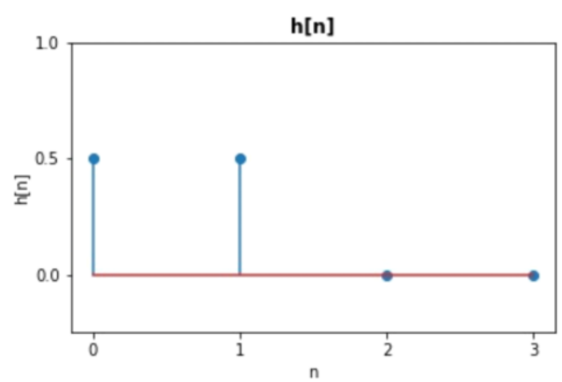
$$\begin{bmatrix} * & & & 0 \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{bmatrix} \begin{bmatrix} * \\ * \\ \vdots \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \\ \vdots \\ * \end{bmatrix} \odot \begin{bmatrix} * \\ * \\ \vdots \\ * \end{bmatrix}$$

1 Circulant Matrices & Convolution

Consider the signal $x[n]$ of length 3 and an impulse response $h[n]$ of length 2. You may assume that they are zero everywhere else.

$$\vec{x} = \begin{bmatrix} -1 & 3 & -2 \end{bmatrix}^T \qquad \vec{h} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}^T \tag{6}$$

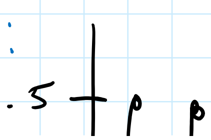




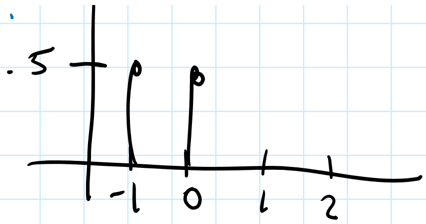
a) What is the convolution $y[n] = x[n] * h[n]$? Also what is the length of this output signal?

$F = 4 \times 4$, $x = 4 \times 1$

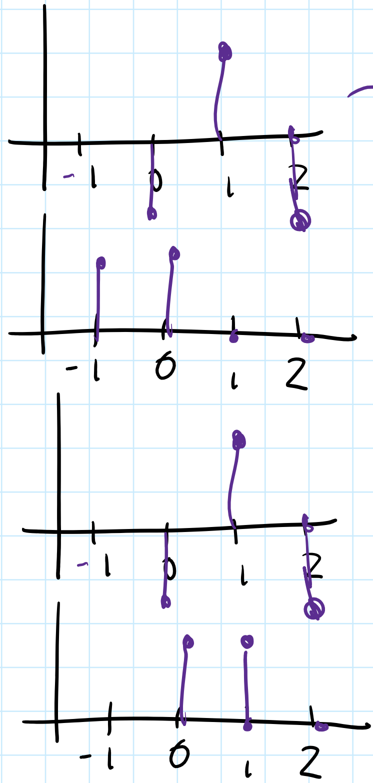
· Flip h:



· flip h:



· line up w/ x:



0th element: $-1 \cdot 0.5 = 0.5$

1st element: $-1 \cdot 0.5 + 2 \cdot 0.5 = 1$

⋮ continue to get

$y = [0.5, 1, 0.5, -1]^T$

d) Since the DFT diagonalizes circulant matrices, lets try to solve for the output signal $y[n]$ using the DFT instead of convolution.

- Step 1: Compute the DFT of $x[n]$ and $h[n]$: $\vec{X} = F\vec{x}$, $\vec{H} = F\vec{h}$.
- Step 2: Take the elementwise product of the DFTs and scale: $\vec{Y} = \sqrt{N}\vec{X} \odot \vec{H}$.
- Step 3: Perform the inverse DFT to get the result $\vec{y} = F^*\vec{Y}$.

$$y = F^* \left(\sqrt{N}\, F \overset{\text{known}}{h}\right) \odot \left(F \overset{\text{known}}{x}\right)$$

$$F^* = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j\frac{2\pi}{4}} & e^{j\frac{4\pi}{4}} & e^{j\frac{6\pi}{4}} \\ 1 & e^{j\frac{4\pi}{4}} & e^{j\frac{8\pi}{4}} & e^{j\frac{12\pi}{4}} \\ 1 & e^{j\frac{6\pi}{4}} & e^{j\frac{12\pi}{4}} & e^{j\frac{18\pi}{4}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$