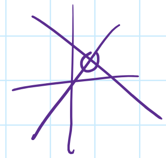
Why linearize?

- Analytic solutions to differential equations
- Predictable stability
- Easy to design controllers for
- Equlibria can be fully characterized

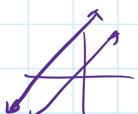
When the system is linear,

$$\dot{x}(t) = \underset{n\times n}{A}\, x(t) + \underset{n\times m}{B}\, u(t)$$

↳ if A is invertible, there is a unique equlibrium point for each input u : $x = -A^{-1}Bu$

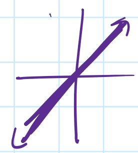


↳ Not invertible: (Rank A < R)



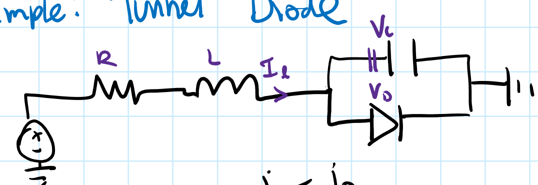
No equlibrium: no solution to $Ax + Bu = 0$

Infinite number of equlibria: $Bu \in Col(A)$



- Isolated equilibria for linear systems do not exist

Example: Tunnel Diode



$$I = C\frac{dV}{dt} = i_L - i_D$$

$$V_L = L\frac{dI}{dt} = -V_C - Ri_L + V_{in} \Rightarrow x(t) = \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix}, \quad u_{in} = V_{in}(t)$$

$$\dot{x}(t) = \begin{bmatrix} \frac{x_2(t) - g(x_1(t))}{C} \\ \frac{1}{L}(-x_1 - Rx_2 + u(t)) \end{bmatrix}$$

Equilibrium at

$$X_{eq}(u) = \left\{ x \,\middle|\, \begin{matrix} \frac{x_2 - g(x_1(t))}{C} = 0 \\ -x_1(t) - Rx_2 + u(t) = 0 \end{matrix} \right\}$$

$$= \left\{ x \,\middle|\, \begin{matrix} x_2 = g(x_1(t)) \\ x_2 = \frac{u - x_1}{R} \end{matrix} \right\}$$

— nonlinear relation: can't predict equlibrium

$$\frac{dV_C}{dt} = \frac{i_C}{C} = 0 \quad \text{at equlibrium}$$

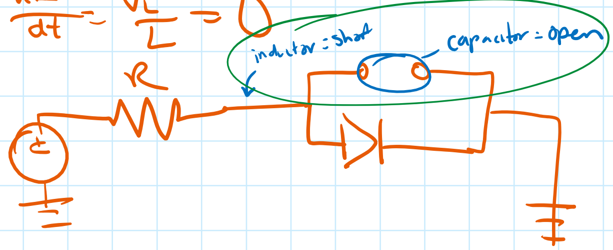
$\frac{dI}{...}$ V_L ... 0 true if V_L, I...

$$\frac{dV_c}{dt} = \frac{i_c}{C} = 0$$

$$\frac{dI}{dt} = \frac{V_L}{L} = 0$$

at equilibrium

true if V_L, I_C are state variables



Linearization

Making a linear approximation of a non-linear system around points of interest

1st order Taylor approximation of $f(x)$:

$$f(x) \approx f(x^*) + \nabla f(x^*)(x - x^*)$$

Jacobian:

$$\nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

If x^* is an equilibrium point, $f(x^*) = 0$ so

$$\frac{d}{dt}(x(t) - x^*) \approx \nabla f(x^*)(x(t) - x^*)$$

follows form $x' = Ax$

Linearization of Inputs

Consider $\dot{x}(t) = f(x(t), u(t))$:

w/ Linearization (x^*, u^*) such that

$$\dot{x}(t) \approx f(x, u) + \nabla_x f(x^*, u^*)(x - x^*) + \nabla_u f(x^*, u^*)(u - u^*)$$

Practice: Jacobian

Calculate Jacobian for $f(x_1, x_2) = \begin{bmatrix} x_1^2 - x_2^2 \\ x_1^2 + x_1 x_2^2 \\ x_1 \end{bmatrix}$

Calculate Jacobian for $f(x_1, x_2) = \begin{bmatrix} x_1^2 + x_1 x_2^2 \\ x_1 \end{bmatrix}$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 & -2x_2 \\ 2x_1 + x_2^2 & 2x_1 x_2 \\ 1 & 0 \end{bmatrix}$$

Practice: Linearization

Let $\frac{\partial^2 y}{\partial t^2} = -\frac{2k}{m}\left(y - X_0 \frac{y}{\sqrt{y^2+a^2}}\right).$

- $x_1 = y$
- $x_2 = \dot{y}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} x_2 \\ -\frac{2k}{m}\left(x_1 - X_0 \frac{x_1}{\sqrt{a^2+x_1^2}}\right) \end{bmatrix}$$

Find equilibrium.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{2k}{m}\left(x_1 - x_0 \frac{x_1}{\sqrt{a^2+x_1^2}}\right) \end{bmatrix}$$

So $x_1 = X_0 \frac{x_1}{\sqrt{a^2+x_1^2}}$

$X_0 = \sqrt{a^2+x_1^2}$

$x_1^2 = X_0^2 - a^2 \Rightarrow$

Compute Jacobian

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{2k}{m}\left(1 - X_0\left((x^2+a^2)^{-3/2} \cdot 2x^2 \cdot \frac{-1}{2} + \frac{1}{\sqrt{x_1^2+a^2}}\right)\right) & 0 \end{bmatrix}$$

plug in $x_1^2 = X_0^2 - a^2$

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$$\dot{\mathring{X}} \approx \begin{bmatrix} 0 & 1 \\ \left(-\frac{2k}{m}\right)\left(\frac{X_0}{a} - 1\right) & 0 \end{bmatrix} X$$

Solving Linearized State Space

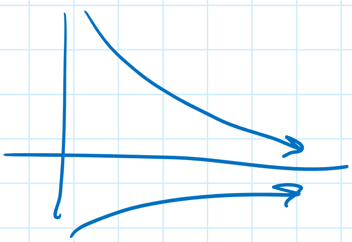
Let there be a 1D state space

$$\dot{x}(t) = \lambda \delta x(t) + b \delta u(t).$$

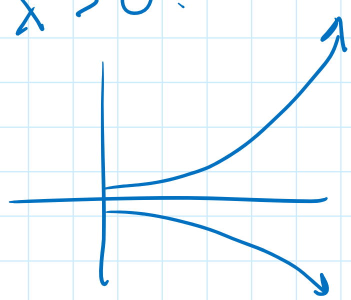
- what happens if $u = 0$ (no control)?
  - ↳ $\partial \dot{x}(t) = \lambda \delta x(t)$

$$\partial x(t) = \partial x(t_0) e^{\lambda(t - t_0)}$$

$\lambda < 0$:



$\lambda > 0$:



- Consider $u \neq 0$:

$$\delta x(t) = e^{\lambda(t-t_0)} \delta x(t_0) + \int_{t_0}^{t} e^{\lambda(t-\tau)} b \delta u(\tau) d\tau$$

If u is constant, $t_0 = 0$:

$$\delta x(t) = e^{\lambda t} \delta x(0) + \frac{b \delta u}{\lambda}\left(e^{\lambda t} - 1\right)$$

If linearizing about a non-equilibrium point:

$$\delta \dot{x}(t) \approx \lambda \delta x(t) + b \delta u(t) + c$$

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$$\delta \dot{x}(t) \approx \lambda \delta x(t) + b \delta u(t) + C$$