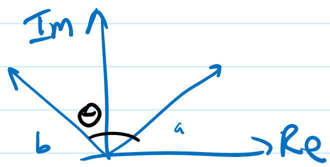
Phasors

Monday, June 29, 2020 10:14 AM

Complex Numbers

$z = a + jb$

- Magnitude: $|z| = \sqrt{a^2+b^2}$
- Phase: $\theta = \angle z = \underline{atan2}(b,a)$
  $\hookrightarrow -\pi \leq atan2(b,a) \leq \pi$



Polar form: $z = |z| e^{j\angle z}$

Derivation: $a = r\cos\theta$, $b = r\sin\theta$ (geometry)

$f(a,b) \to f(r,\theta)$: $z = a + jb$

$$= r\cos\theta + jr\sin\theta$$

$$= r\underbrace{(\cos\theta + j\sin\theta)}_{\text{Euler's Formula}}$$

$$= r e^{j\theta} = |z| e^{j\angle z}$$

Taylor Series derivation:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots$$

So $e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2} + \frac{(j\theta)^3}{3!} \ldots\ldots$ ($j^2 = -1$)

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \ldots\ldots$$
$$+ j(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \ldots$$

$$= \sin\theta + j\cos\theta$$

Example: $\sqrt{j} = (j)^{\frac{1}{2}} = (e^{\frac{\pi}{2}j})^{\frac{1}{2}} = e^{\frac{\pi}{4}j}$

Roots of $z^n = 1$

- Convert 1 using Euler's Formula:

$$1 = 1e^{0i} = 1e^{2\pi i} \ldots = 1e^{2k\pi i}$$

$$(z^n)^{\frac{1}{n}} = (1e^{2\pi i})^{\frac{1}{n}}$$

$$z = e^{2\frac{k}{n}\pi i}$$ for all k such that $\frac{k}{n} < 2\pi$

Phasors

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

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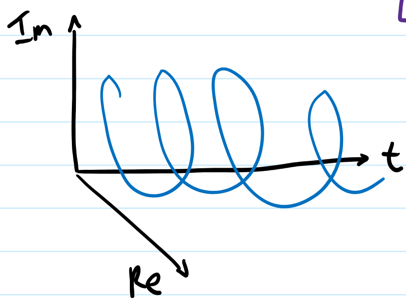
ω ↳ angular frequency (rad/s)



Scaling: $re^{j\cdot 0} \cdot e^{j\omega t}$

Rotating: $1 \cdot e^{j\phi} \cdot e^{j\omega t}$

Linear Combinations of $e^{j\omega t}$

- Sinusoids:

  → $\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$

  → $\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$ =

- Phasor sinusoids:

  → $r\cos(\omega t + \phi) = \frac{1}{2}(re^{j\omega t + \phi} + re^{j\omega t + \phi}$

  $= re^{j\phi} \cdot \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$

- Multi-frequency combinations:

  → $r\cos(\omega_1 t) + r\cos(\omega_2 t) = \frac{1}{2}(e^{j\omega_1 t} + e^{-j\omega_2 t}) + \frac{1}{2}(e^{j\omega_2 t} + e^{-j\omega_2 t})$

  linear combination of $e^{j\omega t}$'s

  ↳ Allows us to stay in Phasor Domain

$\tilde{V}_{in} e^{j\omega t} \rightarrow \boxed{\text{Circuit}} \rightarrow \lambda \tilde{V}_{in} e^{j\omega t}$

Facts in Phasor Domain:

- $Z_R = R$
- $Z_L = j\omega L$
- $Z_C = \frac{1}{j\omega C}$

} imaginary impedance: makes current go out of phase w/ voltage.

⇓

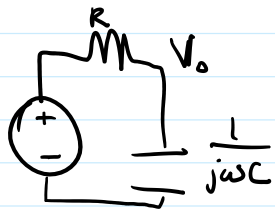
Power through an imaginary impedance is returned

- $Z_C = \frac{1}{j\omega C}$ } go out of phase w/ voltage.

$\Downarrow$

Power through an imaginary impedance is returned

Example: real + imaginary impedance



- $V_o = \lambda V_{in}$ by eigenfunction
- $V_o = V_{in} \frac{\frac{1}{j\omega c}}{\frac{1}{j\omega c}+R}$ by voltage dividor

$$\frac{V_o}{V_{in}} = \frac{1-j\omega RC}{1+(\omega RC)^2}$$

$$= \frac{1}{1+(\omega RC)^2} + j\left(\frac{-\omega RC}{1+(\omega RC)^2}\right)$$

Transfer Function $H(j\omega)$

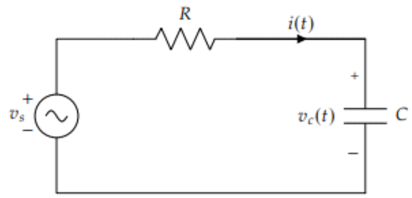
2 Phasor Analysis

Any sinusoidal time-varying function $x(t)$, representing a voltage or a current, can be expressed in the form

$$x(t) = \Re\mathfrak{e}[Xe^{j\omega t}], \tag{1}$$

where X is a time-independent function called the phasor counterpart of $x(t)$. Thus, $x(t)$ is defined in the time domain, while its counterpart X is defined in the phasor domain.

The phasor analysis method consists of five steps. Consider the RC circuit below.



The voltage source is given by

$$v_s(t) = 12\sin\left(\omega t - \frac{\pi}{4}\right), \tag{2}$$

with $\omega = 1 \times 10^3\ \frac{\text{rad}}{\text{s}}$, $R = \sqrt{3}\text{k}\Omega$, and $C = 1\,\mu\text{F}$.

Our goal is to obtain a solution for $i(t)$ with the sinusoidal voltage source $v_s(t)$.

a) **Step 1: Adopt cosine references**

All voltages and currents with known sinusoidal functions should be expressed in the standard cosine format. Convert $v_s(t)$ into a cosine and write down its phasor representation $\tilde{V}_s$.

$\sin\theta = \cos(\theta - \frac{\pi}{2})$

$12\sin(\omega t - \frac{\pi}{4}) = 12\cos(\omega t - \frac{3\pi}{4})$

Phasor: $\tilde{V}_s = V_0 e^{j\phi} = \underline{12e^{-\frac{3\pi}{4}j}}$

time invariant

b) **Step 2: Transform circuits to phasor domain**

The voltage source is represented by its phasor $\tilde{V}_s$. The current $i(t)$ is related to its phasor counterpart $\tilde{I}$ by

$$i(t) = \Re e[\tilde{I} e^{j\omega t}].$$

What are the phasor representations of R and C?

Resistor: $I = \frac{V}{R} \Rightarrow V_R(t) = \cos(\omega t + \phi) = V_0 e^{j\phi} \cdot e^{j\omega t}$

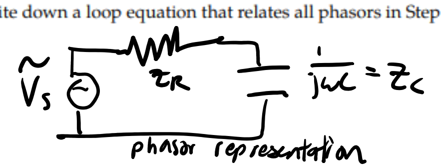
$I_R(t) = \frac{V_0}{R} \cos(\omega t + \phi) = \frac{V_0}{R} e^{j\phi} e^{j\omega t}$

Capacitor: $I = \frac{dQ}{dt} = C \frac{dV}{dt} = \frac{V_0}{j\omega C} \cos(\omega t + \phi) = \frac{V_0}{j\omega C} e^{j\phi} e^{j\omega t}$

c) **Step 3: Cast KCL and/or KVL equations in phasor domain**

Use Kirchhoff's laws to write down a loop equation that relates all phasors in Step 2.

$I_R = I_C$



$V_s(t) - V_R(t) - V_c(t) = 0$

$V_s(t) - R \cdot I_R(t) - \frac{1}{j\omega C} I_c(t) = 0$

$\tilde{V}_s - \tilde{I}_s \left(R + \frac{1}{j\omega C}\right) = 0$

d) **Step 4: Solve for unknown variables**

Solve the equation you derived in Step 3 for $\tilde{I}$ and $\tilde{V}_c$. What are the polar forms of $\tilde{I}$ ($Ae^{j\theta}$, where A is a positive real number) and $\tilde{V}_c$?

$$\tilde{V}_s = \tilde{I}_s\left(R + \frac{1}{j\omega C}\right)$$

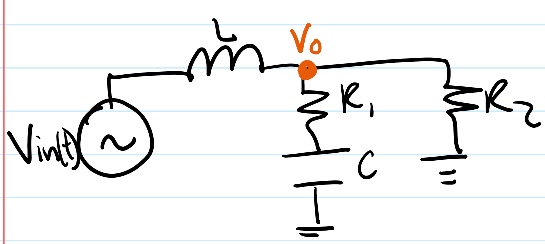
$$\tilde{I}_s = \frac{\tilde{V}_s}{R + \frac{1}{j\omega C}} \cdot \frac{j\omega C}{j\omega C} = \frac{\tilde{V}_s \cdot j\omega C}{Rj\omega C + 1}$$

$$V_c = \tilde{I}_s \cdot \frac{1}{j\omega C}$$

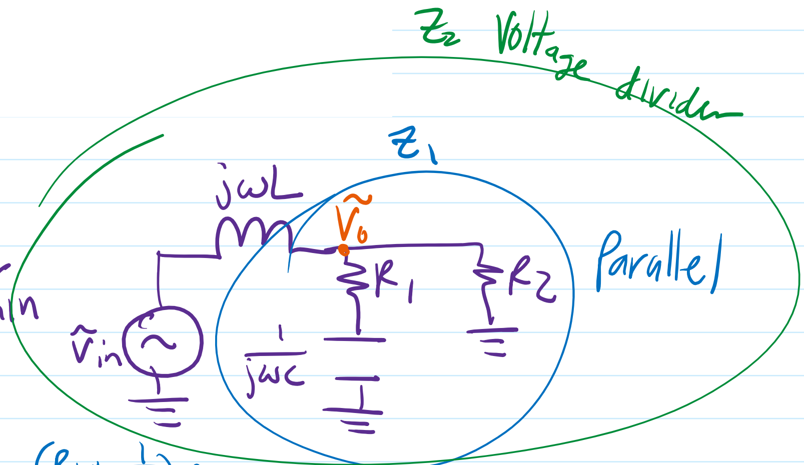
e) **Step 5: Transform solutions back to time domain**

To return to time domain, we apply the fundamental relation between a sinusoidal function and its phasor counterpart. What is $i(t)$ and $v_c(t)$? What is the phase difference between $i(t)$ and $v_c(t)$?

Transfer Function Examples



Phasor Domain $\Longrightarrow$



$$Z_1 = \frac{\left(R_1 + \frac{1}{j\omega C}\right)R_2}{\left(R_1 + \frac{1}{j\omega C}\right) + R_2}$$

$$Z_2 = \frac{Z_1}{Z_1 + j\omega L} = H(j\omega)$$

Magnitude $= |H(j\omega)| \rightarrow$ numpy.absolute

$$= \frac{\sqrt{1^2 + \frac{\omega L^2}{\omega_0} \ldots}}{}$$

Magnitude $|H(j\omega)|$

$$= \frac{\sqrt{1^2 + \frac{\omega_1^2}{\omega_{z}} \cdots}}{\sqrt{1^2 + \frac{\omega_1^2}{\omega_p^2} \cdots}}$$

Phase angle $\measuredangle H(j\omega) \rightarrow$ numpy.angle

$$\frac{e^{j\theta_1}}{e^{j\theta_2}} = e^{j\theta_1 - \theta_2} \text{ so } \measuredangle H(j\omega) = \angle\left(\frac{1+\frac{\omega_1}{\omega_z}\cdots}{1+\frac{\omega}{\omega_p}\cdots}\right)$$

$$= \angle\left(1+\frac{\omega_1}{\omega_z}\cdots\right) - \angle\left(1+\frac{\omega}{\omega_p}\right)\cdots$$

$$= \operatorname{atan2}\left(1+\frac{\omega_1}{\omega_z}\right) - \operatorname{atan2}\left(1+\frac{\omega}{\omega_p}\cdots\right)$$

Output: $V_{out}(t) = |H(j\omega)| \cdot V_{in}(t + \angle H(j\omega))$