Singular Value Decomposition

Definitions:

σ_1 = maximum amplification = $\max \|Av\| = \sqrt{\lambda_{max}(A^*A)}$

σ_2 = 2nd greatest (largest amplification of orthogonal vector to σ_1)

$\vdots$

$\sigma_{r = rank(A)}$ = min. amplification

SVD Abstract Definition:

Let $f: V \rightarrow U$ be a linear map. A SVD of f is:

- an orthonormal basis $\{v_1 \dots v_n\}$ for V
- an orthonormal basis $\{u_1 \dots u_m\}$ for U
- $\sigma_1 \geq \sigma_2 \geq \dots \sigma_r > 0$ for $f(v_i) = \sigma_i u_i$

Facts about f^*f:

i. f^*f is self-adjoint. $(f^*f)^* = f^*f$
ii. The eigenvalues of f^*f are non-negative.
iii. $null(f^*f) = null(f)$.
iv. $rank(f^*f) = rank(f)$ ← by rank-nullity thm.

- $span\{v_{r+1}, \dots v_n\} = Null(A)$
- $span\{u_1 \dots u_r\} = Col(A)$

Building the SVD

Theorem: If $f: V \rightarrow U$ is linear, it has an SVD. (Existence)

Let $\lambda_1 \geq \lambda_2 \dots \geq \lambda_r > 0$ where $r = rank(f)$, with corresponding normalized eigenvectors $v_1 \dots v_r$.

Select a $\sigma_i = \sqrt{\lambda_i}$ and $u_i = \frac{f(v)}{\sigma_i}$.

Repeat for $0 \leq i \leq r$.

1. Compute A^*A
2. Identify r positive eigenvals of A^*A
3. Identify r orthonormal eigenvectors
4. $\sigma_i = \sqrt{\lambda_i}$
5. $u_i = \frac{Av_i}{\sigma_i}$

Example: SVD of identity map

1. Compute $I^*I = II = I$
2. get eigenvectors/values of I^*I:
   all vectors are eigenvectors w/ $\lambda = 1$.
3. Do $u_1 = \frac{Iv}{\sigma_i} = v$

Therefore the SVD of I is any orthonormal basis of $\mathbb{C}^n$

Dimensions

$A = U \Sigma V^*$

A: $m \times n$, U: $m \times m$, Σ: $m \times n$, V^*: $n \times n$ ($U \Sigma V^*$: $m \times n$)

Compact:

$U_r \Sigma_r V_r^* = m \times n$

U_r: $m \times r$, Σ_r: $r \times r$, V_r^*: $r \times n$

(1) Find the eigenvalues λ_i of A^*A and order them such that $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $\lambda_{r+1} = \cdots = \lambda_n = 0$.

(2) Find the orthonormal eigenvectors of A^*A, so that

$$A^*A\vec{v}_i = \lambda_i \vec{v}_i, \quad i = 1, \ldots, r$$

Note that the vectors must be orthonormal, that is $\langle \vec{v}_i, \vec{v}_i \rangle = 1$ and $\langle \vec{v}_i, v_j \rangle = 0$ for $i \neq j$.

(3) Let $\sigma_i = \sqrt{\lambda_i}$ and set

$$\vec{u}_i = \frac{A\vec{v}_i}{\sigma_i}, \quad i = 1, \ldots, r$$

(4) If $r < n$ then we complete the U and V matrices by adding vectors $\vec{u}_{r+1}, \ldots, \vec{v}_m$ and $\vec{v}_{r+1}, \ldots, \vec{v}_n$ to create an orthonormal bases for $\mathbb{R}^m$ and $\mathbb{R}^n$.

Example: Real Numbers

Find SVD of $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$

$A^*A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$

$\lambda_1 = 18, \quad \lambda_2 = 0.$

$V_1 = \text{null}\left(\begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

normalize to get $V_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$V_2 = \text{null}\begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{18} = 3\sqrt{2}. \quad u_1 = \frac{Av_1}{\sigma} = \frac{1}{3\sqrt{2}}\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\frac{1}{\sqrt{2}}$

$= \frac{1}{6}\begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.

Compact Form: $A = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} 3\sqrt{2} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \end{bmatrix}$

→ **More on Compact SVD**

Form: $A = U\Sigma V^T = \begin{bmatrix} U_c & U_2 \end{bmatrix}\begin{bmatrix} \Sigma_c \\ 0 \end{bmatrix} V^T$

$= U_c \Sigma_c V_c^T$

Full Form:

$$\begin{bmatrix} 1/3 & | & | \\ -2/3 & u_2 & u_3 \\ 2/3 & | & | \end{bmatrix}\begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Start with $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, columns e_1, e_2, e_3.

Use Gram-Schmidt on $[u_1, e_1, e_2]$.

$u_2 = e_1 - \text{proj}_{u_1} e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \langle e_1, u_1 \rangle u_1$

* Sometimes this results in a zero vector

Facts about A

- $\text{Rank}(A) = \text{Rank}(A^*A) = $ # of nonzero eigenvectors
- $\text{Null}(A) = \text{Null}(A^*A)$

Complex SVD Example

$A = \begin{bmatrix} 1 & -j \\ 0 & 1 \\ \ldots & 0 \end{bmatrix}$

Should be self adjoint

$A = \begin{bmatrix} 1 & -j \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

self adjoint

$A^*A = \begin{pmatrix} 1 & 0 & 1 \\ j & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -j \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 2 & -j \\ j & 2 \end{bmatrix}$

$\det(A^*A - \lambda I) = \lambda^2 - 4\lambda + 3 = (s-3)(s-1)$

$\lambda_1 = 3, \lambda_2 = 1$

$\text{Null}(A^*A - 3I) = \left(\begin{array}{cc|c} -1 & -j & 0 \\ j & -1 & 0 \end{array}\right) \Rightarrow \begin{bmatrix} 1 \\ j \end{bmatrix}$

$V_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ j \end{bmatrix}$

$V_2 = \begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} j \\ 1 \end{bmatrix}$

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$

$\sigma_2 = \sqrt{\lambda_2} = 1$

$u_1 = \frac{AV_1}{\sigma_1} = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ j \\ 1 \end{bmatrix}$

$u_2 = \frac{AV_2}{\sigma_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ j \end{bmatrix}$

Truncated SVD $\quad A = U_t \Sigma_t V_t^*$

$= \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 \\ \frac{j}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{j}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-j}{\sqrt{2}} \\ \frac{-j}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Full SVD

$\begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & 1 & 0 & 0 \\ \frac{j}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 & 1 & 0 \\ \frac{1}{\sqrt{6}} & \frac{j}{\sqrt{2}} & 0 & 0 & 1 \end{bmatrix}$

Gram-Schmidt: $e_1 - \text{proj}_{u_1} e_1$

$= e_1 - \left(\frac{2}{\sqrt{6}}\right)V$

$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} \\ \frac{j}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{j}{3} \\ -\frac{1}{3} \end{bmatrix}$

normalize to get $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{j}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$

$= \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{j}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-j}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{j}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-j}{\sqrt{2}} \\ \frac{-j}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$