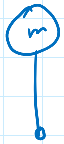
Stability

Tuesday, July 28, 2020 10:15 AM

- Useful operating points in a system may be unstable, and we wish to apply controls to stabilize it.



example: keep pendulum upright

- When using discrete control, the system will never exactly be at equilibrium

Stability in Scalar Case:

Given a discrete time system $x_{k+1} = \lambda x_k + bu_k$:

The system is <u>Stable</u> if the state remains bounded for any initial configuration and any bounded input sequence

↳ if input is finite, output is also finite

- true if $|\lambda| < 1$

The system is <u>unstable</u> if $|x_k| \to \infty$ as $k \to \infty$ for some input sequence

- true if $|\lambda| > 1$

If $|\lambda| < 1$:

$$|x_k| = |\lambda^k x_0 + \sum_{t=0}^{k-1} \lambda^{k-1-t} b u_t|$$

$$\leq |\lambda^k x_0| + |\sum_{t=0}^{k-1} \lambda^{k-1-t} b u_t|$$

$$\leq |\lambda^k x_0| + |b| \sum_{t=0}^{k-1} |\lambda^{k-1-t}| \, |u_t|$$

$$\leq |\lambda^k x_0| + |b| M \sum_{t=0}^{k-1} |\lambda^{k-1-t}|$$

$$\leq |x_0| + |b| M \sum_{t=0}^{k-1} |\lambda|^t$$

$$\leq |x_0| + |b| M \frac{1}{1-|\lambda|}$$

$\Rightarrow$ If $|\lambda| < 1$

$\Rightarrow$ ... remains bounded $\Rightarrow$ Stable

If $|\lambda| = 1$: Marginally Stable

↳ no inputs = stable since $|x_k| = |\lambda^k x_0| = |x_0|$

↳ some inputs = possibly unstable

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· stability properties are unchanged if a constant is added.

Stability in Vector Case

Consider $x[k+1] = Ax[k] + Bu[k]$.

$$x[k] = A^k x[0] + \sum_{t=0}^{k-1} A^{k-1-t} Bu[t]$$

· If A diagonalizable:

$A = V^{-1} \Lambda V$. Transform to V basis:

$z[k+1] = \Lambda z[k] + V^{-1} B u[k]$

↑ collection of eigenvalues, so equivalent to:

$z_i[k+1] = \lambda_i z[k] + V^{-1} b_i u[k]$ for $1 \le i \le n$ (# of eigenvalues).

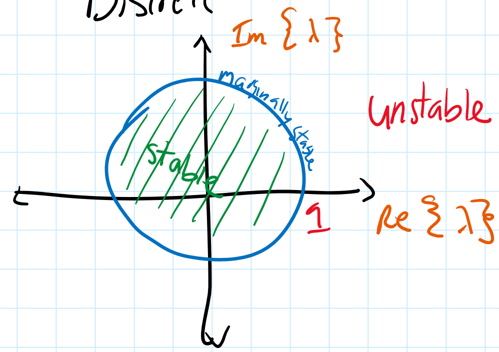
For whole system:

$|\lambda_i| > 1$ for any i, unstable

$|\lambda_i| < 1$ for all i, stable

$|\lambda_i| \le 1$ for all i, marginally stable

Discrete



Non-diagonalizable Stability

* Stability criteria the same as diagonalizable case (all eigenvalues < 1 = stable)

Proof:

• all square matrices A can be upper triangularized such that

$$TAT^{-1} = U = \begin{bmatrix} \lambda_1 & \cdots & * \\ 0 & \ddots & \lambda_n \end{bmatrix}$$

Let $z[k] = Tx[k]$, $x[k] = T^{-1} z[k]$.

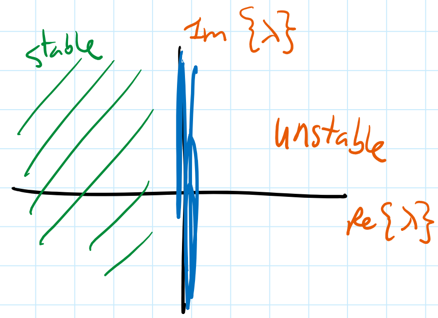
$$z[k+1] = U z[k] + B_{new} u[k]$$

- Due to upper triangular structure, the n^{th} dimension is always scaled by λ_n. If $|\lambda_n| < 1$: bounded

base → $z_{n-1}[k+1] = \lambda_{n-1} z_{n-1}[k] + \underbrace{(* z_n[k] + b_{n-1} u[k])}_{= \hat{u}_{n-1}[k] = \text{bounded}}$

recursive → So if $|\lambda_{n-1}| < 1$, this 2-equation system is bounded.

Continuous



Continuous time stability

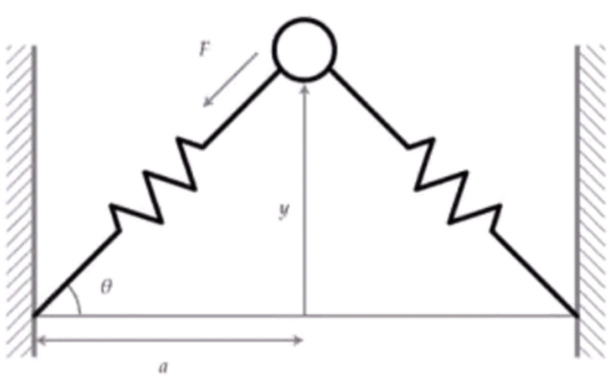
$$|x(t)| = \left| e^{\lambda t} x(0) + \int_0^t e^{\lambda(t-s)} b u(s)\, ds \right|$$

$\lambda = \alpha + j\omega$

- $Re\{\lambda\} < 0 \Rightarrow$ stable
  $Re\{\lambda\} > 0 \Rightarrow$ unstable
  $Re\{\lambda\} = 0 \Rightarrow$ marginally stable

2 Stability in continuous time system

Remember the spring-mass system introduced in Discussion 8A:



We assumed that each spring is linear with spring constant k and resting length X_0. The differential equation modeling this system was $\frac{d^2y}{dt^2} = -\frac{2k}{m}(y - X_0\frac{y}{\sqrt{y^2+a^2}})$. We built a state space model that describes how the displacement y of the mass from the spring base evolves. The state variables were $x_1 = y$ and $x_2 = \dot{y}$. Then we linearized the model around the equilibrium point $(x_1, x_2) = (0, 0)$, assuming $X_0 < a$. The linearized model is presented below.

m : mass

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{2k}{m}\left(1 - \frac{X_0}{a}\right) & 0 \end{bmatrix} x.$$

Compute the eigenvalues of your linearized model. Is this equilibrium stable?

$$(-\lambda)^2 + \frac{2k}{m}\left(1 - \frac{x_0}{a}\right) = 0$$

$$\lambda = \pm\sqrt{\frac{2k}{m}\left(\frac{x_0}{a} - 1\right)}$$

Assuming $x_0 < a$, $\frac{x_0}{a} - 1 < 0$ so λ is imaginary.

$Re\{\lambda\} = 0$ so the system is marginally stable

3 Stability in discrete time system

Determine which values of α and β will make the following discrete-time state space models stable. Assume, α and β are real numbers and $b \neq 0$.

* The discrete-time system

$\Sigma_d: x[t+1] = Ax[t] + Bu[t]$

is stable

$\Leftrightarrow |\lambda_i| < 1$ for all eigenvalues λ_i of A

a)

$$x[t+1] = \alpha x[t] + bu(t)$$

$$|\alpha| < 1$$

b)

rotation matrix

$$\vec{x}[t+1] = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \vec{x}[t] + b\vec{u}(t)$$

$$(\lambda - \alpha)^2 + \beta^2$$

$$\lambda = \alpha \pm j\beta$$

$$|\lambda| = \sqrt{\alpha^2 + \beta^2}$$

$\sqrt{\alpha^2 + \beta^2} < 1$ or $\alpha^2 + \beta^2 < 1$

c)

$$\vec{x}[t+1] = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \vec{x}[t] + b\vec{u}(t)$$

$\lambda = 1$ since A is upper triangular so $\lambda = 1$

so this system is always unstable

Stabilizing Control

Tuesday, July 28, 2020 11:31 PM

* for a motivating example (pendulum), view lecture 7/28.

Transients and Eigenvectors

Let $x_{k+1} = Ax_k + Bu_k$.

Let $A = V \Lambda V^{-1}$ where Λ are eigenvalues of A.

Let $z_k = V^{-1} x_k$. Then $x_k = V z_k$.

Change of basis: $z_{k+1} = V^{-1} x_{k+1} = V^{-1}(Ax_k + Bu_k)$

$$z_{k+1} = \Lambda z_k + V^{-1} B u_k.$$

Consider one row of z such that

$$z_i[k+1] = \lambda_i z_i[k] + b_i u[k].$$

Then $z_i[k] = \lambda_i^k z_i[0] + \sum_{t=0}^{k-1} \lambda_i^{k-1-t} b_i u[t]$

In polar form, $\lambda_i = |\lambda_i| e^{j\omega}$ so

$$\lambda_i^t = |\lambda_i|^t (\underbrace{\cos(\omega t)}_{Re} + j \underbrace{\sin(\omega t)}_{Im}).$$

Since $x[k] = V z[k]$ (lin. combo of $z[k]$), we can plot Re and Im to find the transient (how system evolves over time).

$\hookrightarrow$ natural response (no input)

Continuous Time:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

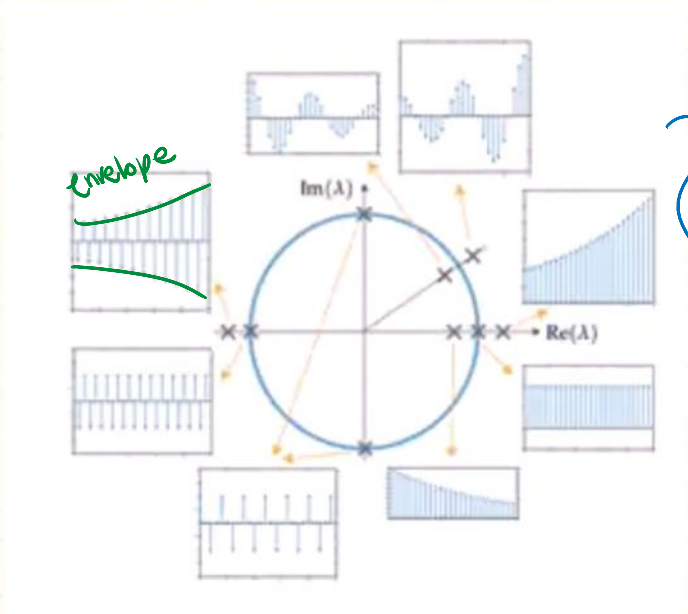
$$z(t) = V^{-1} x(t)$$

$$\dot{z}(t) = \Lambda z(t) + V^{-1} B u(t) \text{ so}$$

$$z_i(t) = e^{\lambda t} z_i(0) + \underline{\text{control}} \text{ stuff..}$$

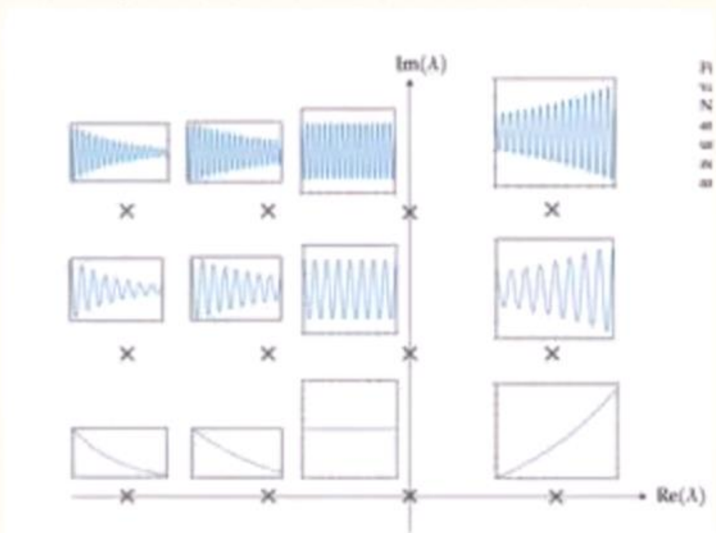
Polar: $z_i(t) = z_i(0) e^{\alpha t} \cos(\omega t) + j z_i(0) e^{\alpha t} \sin(\omega t)$

Discrete-time

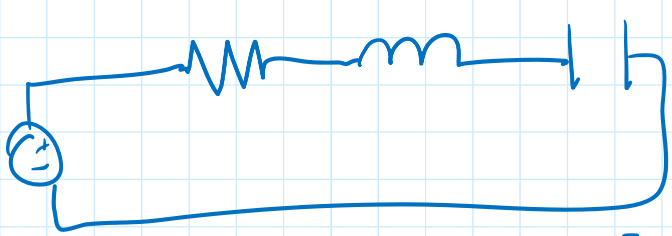


Use eigenvalues to predict behavior

Continuous-time



Example: RLC Circuit



$\dot{x}(t) = \begin{bmatrix} 0 & \frac{1}{C} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t)$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1/C \\ -\frac{1}{L} & R/L \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} u(t)$$

· find eigenvalues

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

$$\lambda = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)}$$

$$= \alpha \pm \sqrt{\alpha^2 - \omega_0^2} \quad \text{where } \alpha = \frac{R}{2L},\ \omega_0 = \frac{1}{\sqrt{LC}}.$$

If $\alpha > \omega_0$, e-values are real and negative.
If $\alpha < \omega_0$, $\lambda = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$ ← decaying oscillation

Feedback

• We want to reach a target x (e.g. 0) from any config in

$$x_{k+1} = Ax_k + Bu_k.$$

· Define Control Policy (a function of state x):

$$u_k = -Kx_k, \quad K \in \mathbb{R}^{m \times n}$$

Such that the system is stable.

$$x_{k+1} = (A - BK)x_k$$

Choose eigenvalues of $(A-BK) < 1$ and have desired transient behavior.

2 Eigenvalues Placement in Discrete Time

Consider the following linear discrete time system

$$\vec{x}[t+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[t] + \vec{w}[t] \quad (1)$$

a) Is this system controllable?

$$C = [B \quad Ab] = \left[\begin{matrix} 1 \\ 0 \end{matrix} \quad \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Controllable

b) Is the linear discrete time system stable?

$$(\lambda)(-1-\lambda) - 2 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$$\lambda = 2, -1$$

c) Derive a state space representation of the resulting closed loop system using state feedback of the form $u[t] = [k_1 \quad k_2]\, \vec{x}[t]$

$$x[t+1] = Ax[t] + Bu[t]$$

$$= Ax[t] + BKx[t]$$

$$= (A + BK)\, x[t]$$

$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}[k_1 \quad k_2]$$

$$= \begin{bmatrix} k_1 & 1+k_2 \\ 2 & -1 \end{bmatrix}$$

d) Find the appropriate state feedback constants, k_1, k_2 in order the state space representation of the resulting closed loop system to place the eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$

$$(k_1 - \lambda)(-1-\lambda) - 2(1+k_2) = 0$$

$$\lambda^2 + \lambda - k_1\lambda - k_1 - 2 - 2k_2 = 0$$

$$\lambda = \frac{k_1 - 1 \pm \sqrt{(1-k_1)^2 + 4(2+2k_2)}}{2}$$

If $\lambda = \pm\frac{1}{2}$, $k_1 - 1 = 0$ and $\sqrt{(1-k_1)^2 + 4(2+2k_2)} = 1$

$k_1 = 1$, $\sqrt{4(2+2k_2)} = 1$

$$4(2+2k_2) = 1$$

$$2 + 2k_2 = \frac{1}{4}$$

$$1 + k_2 = \frac{1}{8}$$

$$2 + 2k_2 = \frac{1}{4}$$

$$1 + k_2 = \frac{1}{8}$$

$$k_2 = -\frac{7}{8}$$