Control Theory: the study of systems which has free inputs

- cruise control
- path tracking

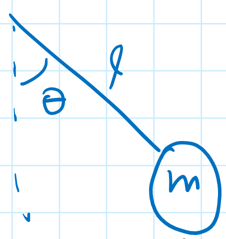
Questions:

- How do we model systems mathematically?
- How can we use data to find parameters?
- How does a system evolve over time?
- Is the system inclined to come to rest?
- Can we change properties of a system by choice of control?

State Space Representation

A way to represent how a model evolves over time
↳ use diff eq's to model systems

Example: Simple Pendulum



$$m\ell\,\ddot{\theta}(t) = -\overbrace{k\ell\,\dot{\theta}(t)}^{\text{angular velocity}} - \overbrace{mg\sin\theta(t)}^{\text{gravity}} + \overbrace{\tau_{in}(t)}^{\text{initial torque}}$$

$$\Downarrow$$

$$\dot{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}' = \begin{bmatrix} x_2(t) \\ -\frac{g}{\ell}\sin(x_1) - \frac{k}{m}x_2(t) + \frac{\tau_{in}}{m\ell}(t) \end{bmatrix}$$

* $x(t)$ fully represents the state of this system - time evolution is a function of these variables

- Since $x \in \mathbb{R}^2$, $\mathbb{R}^2$ is the state space
- Since $\tau_{in}, \theta \in \mathbb{R}$, $\mathbb{R}$ is the control space

Standard form: $\dot{x}(t) = f(\underbrace{x(t)}_{\substack{\text{state space} \\ (\mathbb{R}^n)}}, \underbrace{u(t)}_{\substack{\text{control space} \\ (\mathbb{R}^m)}})$

Equilibrium Point: (state doesn't change with time)

($\mathbb{R}^n$) ($\mathbb{R}^m$)

Equilibrium Point: (state doesn't change with time)

$$(x_{eq}, u_{eq}) \in X \cdot U \text{ such that } f(x_{eq}, u_{eq}) = 0 \Rightarrow \frac{d\vec{x}}{dt} = 0$$

For pendulum: Find x_{eq}, τ_{eq} for

$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ -\frac{g}{\ell}\sin x_1(t) - \frac{k}{m}x_2(t) + \frac{\tau_{in}(t)}{m\ell} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Downarrow$$

$$x_{eq} = 0, \quad \tau_{in}(t) = \frac{g}{\ell}\sin(x_{eq_1}(t))$$

$$\Downarrow$$

$$X_{eq} = \{ (n\pi, 0) \} / n \in \mathbb{N} \}$$

unstable equilibrium — stable equilibrium

Another example: RLC Circuit

Standard form: $x(t) := \begin{bmatrix} V_C(t) \\ i_L(t) \end{bmatrix} := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$$\dot{x}(t) = \begin{bmatrix} \frac{1}{C} \cdot x_2(t) \\ -\frac{1}{L}x_1(t) - \frac{R}{L}x_2(t) + \frac{1}{L}V_{in}(t) \end{bmatrix}$$

Equilibrium: Find $\overset{\begin{bmatrix} V(t) \\ I(t) \end{bmatrix}}{x_{eq}}$, $\overset{V_{in}}{u_{eq}}$ such that $\dot{x}_{eq}(t) = 0$

$\hookrightarrow \frac{1}{C}x_{eq_2} = 0 \Rightarrow x_{eq_2} = 0$

$$\cancel{-\frac{1}{L}}x_{eq_1} - \cancel{\frac{R}{L}x_{eq_2}} + \cancel{\frac{1}{L}}u_{eq} = 0$$

$$x_{eq_1} = u_{eq}$$

$$\parallel \qquad \parallel$$

$$V_C = V_{in}$$