System ID

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★ How do we learn the parameters of a system model from observation?

$$X_{k+1} = A x_k + B u_k$$

find A, B using least squares

. State Observations: $X_1 = a X_0 + b u_0 + e_0$

$$\vdots$$

$$x_{k+1} = a x_k + b u_k + e_k$$

minimize the error for system

$$\begin{bmatrix} x_1 \\ \vdots \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} x_0 & u_0 \\ \vdots & \vdots \\ x_k & u_k \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} e_0 \\ \vdots \\ e_k \end{bmatrix}$$

Using the least squares solution:

$$\begin{bmatrix} a \\ b \end{bmatrix}^* = \left(\begin{bmatrix} x & u \\ \vdots & \vdots \end{bmatrix}^T \begin{bmatrix} x & u \\ \vdots & \vdots \end{bmatrix} \right)^{-1} \begin{bmatrix} x & u \\ \vdots & \vdots \end{bmatrix}^T \begin{bmatrix} x_0 \\ \vdots \\ x_{k+1} \end{bmatrix}$$

$$(D^T D)^{-1} D^T y$$

1 System Identification and Linear Control

A scalar discrete-time system has the following dynamics:

$$x(t+1) = \lambda x[t] + g(u[t]),$$

where $g : \mathbb{R} \to \mathbb{R}$ not necessarily linear.

a) If g is approximated to order 2 around the operating point $u^* = 0$, so that

$$x(t+1) \approx \lambda x[t] + \beta_0 + \beta_1 u[t] + \beta_2 u^2[t],$$

what should β_0, β_1, and β_2 be?

Use Taylor approximation:

$$g(u) \approx g(u[0]) + \frac{dg}{du} u + \frac{1}{2}\frac{d^2 g}{du^2} u^2$$

so
$$\beta_0 = g(u[0])$$
$$\beta_1 = \frac{dg}{du}$$
$$\beta_2 = \frac{d^2 g}{du^2} \cdot \frac{1}{2}$$

b) Suppose that $x[0] = 0$. We apply a sequence of inputs

$$\vec{u} = (u[0], u[1], \ldots, u(N-1))$$

and observe states $x[1], x[2], \ldots, x[N]$. Derive the least-squares estimates of λ, β_0, β_1, and β_2.

System: $x[n+1] = \lambda x[n] + \beta_0 + \beta_1 u[n] + \beta_2 u^2[n] + e[n]$ error

We know that for $y = Dp$,

$$p = (D^T D)^{-1} D^T y$$

Let $p = \begin{bmatrix} \lambda \\ \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$

Then $y = \begin{bmatrix} x[1] \\ \vdots \end{bmatrix} = \begin{bmatrix} x[0] & 1 & u[0] & u^2[0] \\ & & \vdots & \end{bmatrix} \begin{bmatrix} \lambda \\ \beta_0 \\ \vdots \end{bmatrix}$

Then $$y = \begin{bmatrix} x[1] \\ \vdots \\ x[N] \end{bmatrix} = \begin{bmatrix} x[0] & 1 & u[0] & u^2[0] \\ & \vdots & & \\ x[N-1] & 1 & u[N-1] & u^2[N-1] \end{bmatrix} \begin{bmatrix} \lambda \\ \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}.$$

2 System Identification

Let's now look at how System Identification works in the vector case. Again you are given an unknown discrete-time system. We don't know its specifics but we know that it takes one scalar input and as two observable states.

We would like to find a linear model of the form

disturbance

$$\rightarrow \quad \vec{x}[t+1] = A\vec{x}[t] + B\underline{u[t]} + \underline{\vec{w}[t]},$$

where $\vec{w}[t]$ is an error term due to unseen distrubances and noise, $u[t]$ is a scalar input, and

$$2\times 2 \quad \underline{A = \begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix}}, \quad \underline{B = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}}, \quad \underline{\vec{x}[t] = \begin{bmatrix} x_0[t] \\ x_1[t] \end{bmatrix}}.$$

To identify the system parameters from measured data, we need to find the unknowns: a_0, a_1, a_2, a_3, b_0 and b_1, however, you can only interact with the system via a blackbox model. The model allows you to view the states $\vec{x}[t] = \begin{bmatrix} x_0[t] & x_1[t] \end{bmatrix}^T$ and it takes a scalar input $u[t]$ that allows the system to move to the next state $\vec{x}[t+1] = \begin{bmatrix} x_0[t+1] & x_1[t+1] \end{bmatrix}^T$.

a) **Write scalar equations for the new states, $\underline{x_0[t+1]}$ and $\underline{x_1[t+1]}$ in terms of $\underline{a_i, b_i}$, the states $\underline{x_0[t], x_1[t]}$, and the input $\underline{u[t]}$.** Here, assume that $\underline{\vec{w}[t] = \vec{0}}$ (i.e. the model is perfect).

2 equations :

$$x_0[t+1] = (a_0 x_0[t] + a_1 x_1[t]) + b_0 u[t] + w_0[t]$$
$$x_1[t+1] = (a_2 x_0[t] + a_3 x_1[t]) + b_0 u[t] + w_0[t]$$

b) Now we want to identify the system parameters. We observe the system at the initial state $\vec{x}[0] = \begin{bmatrix} x_0[0] \\ x_1[0] \end{bmatrix}$, input $u[0]$ and observe the next state $\vec{x}[1] = \begin{bmatrix} x_0[1] \\ x_1[1] \end{bmatrix}$. We can continue this for an m long sequence of inputs.

What is the minimum value of m you need to identify the system parameters?

There are 6 unknowns, so we need 6 equations.
For each time step, we get 2 equations, so $m = 3$

To identify the system we need to set up an approximate (because of potential disturbances) matrix equation

$$D\vec{p} \approx \vec{y}$$

using the observed values above and the unknown parameters we want to find. Suppose you are given the form of D in terms of some of the observed data:

$$\Rightarrow \quad D = \begin{bmatrix} x_0[0] & x_1[0] & u[0] & 0 & 0 & 0 \\ x_0[1] & x_1[1] & u[1] & 0 & 0 & 0 \\ x_0[2] & x_1[2] & u[2] & 0 & 0 & 0 \\ x_0[3] & x_1[3] & u[3] & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0[0] & x_1[0] & u[0] \\ 0 & 0 & 0 & x_0[1] & x_1[1] & u[1] \\ 0 & 0 & 0 & x_0[2] & x_1[2] & u[2] \\ 0 & 0 & 0 & x_0[3] & x_1[3] & u[3] \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_0 \\ a_2 \\ a_3 \\ b_1 \end{bmatrix} = \begin{bmatrix} x_0[0] \\ x_0[1] \\ x_0[2] \\ x_0[3] \\ x_1[0] \\ x_1[1] \\ x_1[2] \\ x_1[3] \end{bmatrix}$$

For this D, **what are $\vec{y}$ and the unknowns $\vec{p}$ so that $D\vec{p} \approx \vec{y}$ makes sense?** Tell us what the components of these vectors are, written in vector form.