Math 54 Study Guide

Fall 2019 Wehrheim – Created by Ben Cuan

Part 1: Overview

I. Definitions

- L.I. of vectors/functions
- Span of a set of vectors
- Vector Space (9 conditions)
- Subspace (3 conditions)
- Basis of Vector Space
- Linear Transformation (3 conditions)
- One-to-one
- Onto
- Range of L.T.
- Kernel of L.T.
- Column Space of matrix
- Null Space of matrix
- Inverse of matrix
- Dimension of vector space
- Eigenvalue/Eigenvector of matrix
- Diagonalizable matrix
- Coordinate vector $[x]_B$
- Change of coordinates matrix $P_{C \leftarrow B}$ (and how to compute)
- Matrix relative to bases $\mathcal{B}$, C
- Matrix $[T]_{\mathcal{B}}$
- Orthogonal
- Inner Product (4 conditions)
- Norm (4 conditions)
- Orthonormal
- Matrix exponential (and formula for y_h given $e^{At} = \Phi(t)$) (and properties)
- Wronskian (implications for solutions to ODE ; linear independence)
- Fundamental Matrix / fundamental solution set
- Superposition
- Determinant of matrix (implication in IMT/invertibility) (+properties)
- Adjugate matrix

II. Theorems and Proofs

- Invertible Matrix Theorem (onto vs one to one)
- Solution Principle for linear problems (prove)
- Solution Principle for ODE's (prove)
- Variation of Parameters Ansatz (prove)
- Wronskian Proof
- Rank Theorem

III. Solving Problems

- Determine if 2 vectors L.I.
- Find determinant of matrix (2 ways)
- Determine if $V \in H$
- Find missing value in matrix for ∞, unique, no solutions
- Can cols of $m \times n$ matrix A span $\mathbb{R}^m$ if $m<n$, $m=n$, $m>n$?
- Possible echelon forms of $m \times n$ matrix
- find missing value in matrix to make it invertible
- Prove existence/uniqueness of diagonalizable matrices
- find a basis and dimension for subspace spanned by S
- find $[x]_{\mathcal{B}}$
- find M relative to $\mathcal{B}$, C
- find $[T]_{\mathcal{B}}$
- find $P_{C \leftarrow \mathcal{B}}$
- Diagonalize a matrix (complex?)
- Invert a matrix
- Find e-vals/e-vecs (Undetermined coefficients)
- Find general solution to 2nd order ODE (real vs complex method)
- Find general solution given linear transformation, kernel, point
- Use Wronskian to determine linear independence
- Solve $x' = Ax$ (2 methods)
- Find A^n
- Find A for $x' = Ax$ given a solution set
- Show $H = \{v \to V \mid \text{condition}\}$ is a subspace
- Show V is a subspace of H
- Prove that a linear transformation is an isomorphism
- Find e^{At}
- Find Fourier series / sin/cos series
- Find general + unique solutions to PDE's
    - a. homogenous
    - b. mixed boundary conditions
    - c. nonhomogenous boundary conditions
    - d. nonhomogenous PDE
    - e. multidimensional
    - f. heat equation
    - g. wave equation
    - h. Laplace equation

Part 2: Details for Definitions

I. Definitions

Vector Spaces

"Definition": Set of objects w/ well-defined addition and multiplication

Formal Definition: follow 10 axioms

1. The sum $u + v$ is in V. we can add...
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$ } reasonably
4. There exists a vector $\mathbf{0} \in V$ such that $\mathbf{0} + u = u + \mathbf{0} = u$ for every vector $u \in V$.
5. ~~For every vector $u \in V$, there exists a vector, denoted $-u$, such that $u + (-u) = -u + u = \mathbf{0}$.~~ follows from other axioms if $-u = (-1)u$
6. The scalar product $c \cdot u$ is in V. we can scale...
7. $c \cdot (u + v) = (c \cdot u) + (c \cdot v) = c \cdot u + c \cdot v$
8. $(c + d) \cdot u = (c \cdot u) + (d \cdot u)$
9. $c \cdot (d \cdot u) = (cd)u$
10. $1 \cdot u = u$ } reasonably

A set of vectors $\{v_1, v_2 \ldots v_n\}$ is linearly dependent if one or more vectors $v_i \in \text{span}(\{v_1 \ldots v_{i-1}, v_{i+1} \ldots v_n\})$.

The span of $\{v_1 \ldots v_n\} \in V$ is the set of all possible linear combinations of the vectors $\cong \{c_1 v_1 + c_2 v_2 \ldots + c_n v_n \mid c_1, c_2 \ldots c_n \in \mathbb{R}\}$

A set of vectors spans V if all possible vectors in $V \in \text{span}(\{v_1 \ldots v_n\})$. $\text{span}(\{v_1 \ldots v_n\}) = V$

A set of vectors $\mathcal{B}$ is a basis for a vector space V if:

i. all vectors in $\mathcal{B}$ are linearly independent, and
ii. $\mathcal{B}$ spans V.

$H \subset V$ if:

i. $0 \in H$
ii. $\underline{h}_1 + \underline{h}_2 \in H$ for $\underline{h}_1, \underline{h}_2 \in H$
iii. $c\underline{h}_1 \in H$ for $h_1 \in H$, $c \in \mathbb{R}$
iv. If $\underline{h}_1 \in H$, $\underline{h}_1 \in V$

$T: V \to W$ is linear if:

i. $T(\underline{0}) = \underline{0}$
ii. $T(\underline{u}) + T(\underline{v}) = T(\underline{u} + \underline{v})$ given $\underline{u}, \underline{v} \in V$
iii. $T(c\underline{u}) = cT(\underline{u})$ given $\underline{u} \in V$

$T: V \to W$ is one-to-one when $T(\underline{u}) = T(\underline{v})$ iff $\underline{u} = \underline{v}$
(all inputs map to a unique output)

$T: V \to W$ is onto when there exists a vector $\underline{v} \in V$ such that $T(\underline{v}) = \underline{w}$ for all $\underline{w}$ in W.

The kernel of $T: V \to W$ is the entire set of vectors $\underline{v} \in V$ such that $T(\underline{v}) = \underline{0}$.

The range of $T: V \to W$ is the entire set of vectors $\underline{w} \in W$ that result from $T(\underline{v}) = \underline{w}$ for all values of $\underline{v} \in V$.

The null space of an $m \times n$ matrix A is equivalent to the kernel of the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ represented by A.

The column space of an $m \times n$ matrix A is equivalent to the range of the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ represented by A.

A matrix C is the inverse of $n \times n$ matrix A if $CA = AC = I_n$ where I_n is the $n \times n$ identity matrix.

Dimension of vector space: the number of (linearly independent) vectors needed to form a basis of the vector space. if V cannot be spanned by a finite set, then V is infinite dimensional.

A vector $\underline{x}$ is an eigenvector of $n \times n$ matrix A with eigenvalue λ if $A\underline{x} = \lambda \underline{x}$, $\lambda \in \mathbb{R}$. $A\underline{x} \in \text{Span } \underline{x}$

$n \times n$ matrix A is diagonalizable if A is similar to a diagonal matrix D (D has all 0 entries except on diagonals). i.e. $A = PDP^{-1}$ for some P invertible

Coordinate vector $[x]_\mathcal{B}$ for x relative to $\mathcal{B}$ basis of vector space V is equivalent to the vector of weights $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ that allow $\underline{x}$ to be written as a linear combination of the basis vectors of $\mathcal{B}$ $b_1 \ldots b_n$.

$$[x]_\mathcal{B} = \text{solution to } [\underline{b}_1 \ldots \underline{b}_n]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \underline{X}$$

Change of Coordinates Matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ from basis $\mathcal{B}$ to basis $\mathcal{C}$ in V is the matrix

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \left[[\underline{b}_1]_\mathcal{C} \quad [\underline{b}_2]_\mathcal{C} \ldots [\underline{b}_n]_\mathcal{C} \right]$$

Where $\underline{b}_1 \ldots \underline{b}_n$ are the basis vectors for basis $\mathcal{B}$.

Matrix M for $T: V \to W$ relative to bases $\mathcal{B}$, $\mathcal{C}$ in vector spaces V, W

$$M = \left[[T(\underline{b}_1)]_\mathcal{C} \ldots [T(\underline{b}_n)]_\mathcal{C} \right]$$

the matrix where the transformation T is applied to the basis vectors of $\mathcal{B}$ $\underline{b}_1 \ldots \underline{b}_n$ then converted to the basis $\mathcal{C}$.

Matrix $[T]_\mathcal{B}$ for $T: V \to V$ relative to basis $\mathcal{B}$ in vector space V

$$[T]_\mathcal{B} = \left[[T(\underline{b}_1)]_\mathcal{B} \ldots [T(\underline{b}_n)]_\mathcal{B} \right]$$

the matrix when T is applied to the basis vectors $\underline{b}_1 \ldots \underline{b}_n$ of $\mathcal{B}$ then converted to the basis $\mathcal{B}$.

Inner Product on a vector space V is a function that operates on a pair of vectors and results in a real number

$U \cdot V$ in $\mathbb{R}^n$ or $\langle f, g \rangle = \int_a^b f(t)g(t)\,dt$ in $C^\infty[a,b]$

Must satisfy:

- $\langle U, V \rangle = \langle V, U \rangle$ symmetry $\quad \int UV = \int VU$
- $\langle U+V, W \rangle = \langle U+W \rangle + \langle V+W \rangle$ linearity in entries $\quad \int (U+V)W = \int UW + \int VW$
- $\langle cU, V \rangle = c\langle U, V \rangle$ linearity in entries $\quad \int (cU)V = c\int UV$
- $\langle U, U \rangle \geq 0$ and $\langle U, U \rangle = 0$ iff $U = 0$ $\quad \int U^2 = 0$ iff $U = 0$

inner product

inner product space is a vector space that includes an inner product

Norm (length) of a vector $\|V\| = \sqrt{\langle V, V \rangle} = \sqrt{V_1^2 + \dots + V_n^2}$

- $\|cV\| = |c|\,\|V\|$
- $\|V+W\| \leq \|V\| + \|W\|$
- $\|V\|^2 = \langle V, V \rangle$
- $\|V\| = 0 \Leftrightarrow V = 0$

Orthonormal: vectors in set are orthogonal and have norm of 1

- A set with the 0 vector can be orthogonal, but can never be orthonormal or linearly independent
- Make vectors normal by dividing by their norms

Matrix: $e^{A} = I + At + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 \ldots + \frac{t^n}{n!}A^n$

★ If $\underline{X}(t)$ = fundamental matrix for $x' = Ax$, then $\boxed{e^{At} = \underline{X}(t)\,\underline{X}^{-1}(0)}$

Properties of Matrix Exponential

1. $e^{A0} = \underline{I}$
2. $e^{A(t+s)} = e^{At}e^{As}$
3. $(e^{At})^{-1} = e^{-At}$
4. $\frac{d}{dt}e^{At} = Ae^{At}$ so always solves $x' = Ax$
5. $e^{rI_n t} = e^{rt}I_n$
6. $e^{A+B} = e^{A}e^{B}$ if $AB = BA$

Two functions are linearly independent if:

- for every $c \in \mathbb{R}$ have $y_1(t) \neq c y_2(t)$ for some t
- $\frac{d}{dt}\frac{y_2(t)}{y_1(t)} \neq 0$ for some t
- $\frac{y_2(t_0)}{y_1(t_0)} \neq \frac{y_2(t_1)}{y_1(t_1)}$ for some t_0, t_1 — same
- $\det\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1(t_1) & y_2(t_1) \end{bmatrix} \neq$ for some t_0, t_1 — same
- $W = \det\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} \neq 0$ for some t

Only work if y_1, y_2 are Solutions to the same differential equation

$\underbrace{[x_1 \; x_2 \ldots]}_{\text{fundamental matrix}} \underline{C} = \underline{X}_p + \underline{X}_0$

"Superposition Principle" ⇒ making good use of linearity
combining the properties of linearity together

Superposition Principle for subspaces:

If $H \overset{\text{subspace}}{<} V$:

- $0 \in H$
- If $v_1 \ldots v_p \in H$, $c_1 \ldots c_p \in \mathbb{R}$ ⇒ $\underbrace{c_1v_1 + \ldots + c_pv_p}_{\text{(linear combination)}} \in H$

∴ $v_1 \ldots v_p \in H$ then $\text{Span}\{v_1 \cdots v_p\} \subset H$

Superposition Principle for Linear Combinations

- $T(0) = 0$
- $T(c_1v_1 \ldots + c_pv_p) = c_1T(v_1) + \ldots + c_pT(v_p)$

↳ T(linear combo of V...) = linear combo of T(V...)

If $T(v_1) \ldots T(v_p)$ is known, then we know how to find $T(v)$ for any $v \in \text{Span}(v_1 \ldots v_p)$!

Part 3: Details for Proofs

The Invertible Matrix Theorem (§2.3)

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.   l. $\det A \neq 0$

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.   "uniqueness"

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.   "existence"

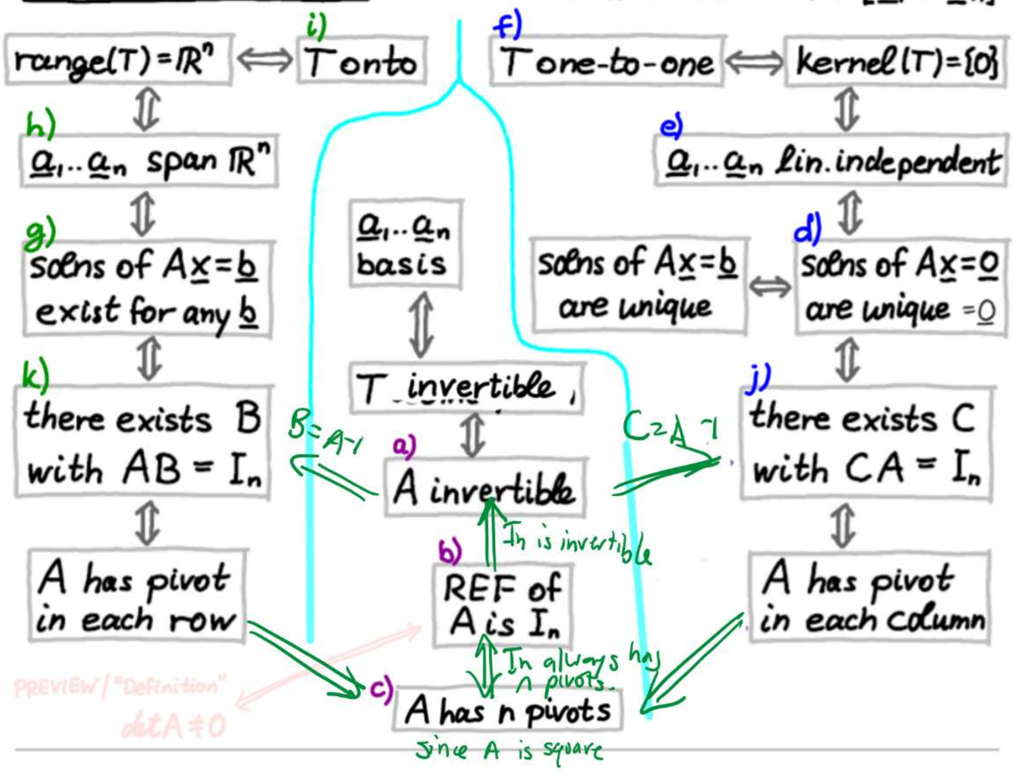
h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

§2 Theorems 5, 7, 8: $T: \mathbb{R}^n \to \mathbb{R}^n$ with matrix $A = [\underline{a}_1 \dots \underline{a}_n]$



Solution Principle

→ Given: linear transformation $T: V \to W$, target $b \in W$

Want: solutions $x \in V$ of $T(x) = b$

★ General solution / solution set: $\{x \in V \mid T(x) = b\}$

Examples:

"all of the solutions"

$T = \frac{d}{dt} : C^\infty \to C^\infty$, $b = 3t^2$ has general solution $\{f \in C^\infty \mid \frac{d}{dt} f = 3t^2\}$

Solution: $f = t^3 + C$

To prove general solution: (example)

Prove If $T(\underline{x}) = \underline{y}$ then $\underline{x} = \underbrace{\underline{a} + r\underline{b}}_{\text{a line}}$ for some $r \in \mathbb{R}$

↑output

★ Use linearity:

If $T(\underline{z}) = \underline{0}$ then $\underline{z} = r\underline{b}$ for $r \in \mathbb{R}$

Also, $T(\underline{x}) = \underline{y}$

then $T(\underline{x} - \underline{y}) = \underline{y} - \underline{y}$.

$T(\underline{x} - \underline{y}) = \underline{0}$

Solution Principle for linear problems

In order to get the set of all solutions you get <u>one</u> solution and then add the kernel / null space

↑ set of $\{\underline{v} \in V \mid T(v) = 0\}$; subspace of V

sits in the domain

$$\left\{\begin{matrix}\text{general solution:} \\ T(V) = b\end{matrix}\right\} = \{V_p + k \mid k = \text{kernel}(T)\}$$

Prove uniqueness of ODE solution

Let there be another possible solution x for the ODE.

$T[x] - T[y_p] = 0$ since they are equal.

By linearity, $T[x - y_p] = 0 = T[y_n]$

Therefore, $x - y_p \in$ Kernel(T) so there exists some values $c_1 \dots c_n \in \mathbb{R}$

for $x - y_p = c_1 f_1 + c_2 f_2 \dots$

Therefore, $x - y_p = y_n$

$x = y_p + y_n$ for all possible solutions x.

Variation of Parameters: if $\underline{f}$ is not e^{rt}, sin/cos, or polynomial

$$x_p = \underline{X}(t) \int \underline{X}^{-1}(t) \underline{f}(t)\,dt$$

Proof of x_p equation:

Definition of fundamental matrix: $\underline{x} = \underline{X}\underline{c}(t)$

$$x' = \underline{X}' c + \underline{X} c'$$

Use $\underline{X}' = A\underline{X}$

$$x' = A\underline{X} c + \underline{X} c'$$

Use original equation $\underline{x}' = A\underline{x} + f(t)$

By coefficient matching, $\underline{x} = \underline{X}c$ and $\underline{X}c'$

Solve for c' : $c'(t) = \underline{X}^{-1}(t) f(t)$

Integrate: $c(t) = \int_0^t \underline{X}^{-1}(s) f(s)\,ds$

Plug into original $\underline{x} = \underline{X}c$ to get

$$\underline{X} = \underline{X} \int_0^t \underline{X}^{-1}(s) f(s)\,ds$$

let $y'' + p(t)y'(t) + q(t)y(t) = 0 \quad (*) \Rightarrow y(t) = \frac{-(y''(t) + p(t)y'(t))}{q(t)}$

If y_1, y_2 are solutions to $(*)$ find $W'(t)$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$$

$$W' = (\cancel{y_1'y_2'} + y_1y_2'') - (y_1''y_2 + \cancel{y_1'y_2'})$$

$$= y_1y_2'' - y_1''y_2$$

$$= -y_1(py_2' + qy_2) + y_2(py_1' + qy_1)$$

$$= \cancel{-qy_1y_2} - py_1y_2' + py_1'y_2 + \cancel{qy_1y_2}$$

$$= py_1'y_2 - py_1y_2' = pW$$

Deduce the first order DE satisfied by $W(t)$

$$W' + pW = 0$$

$$\frac{dW}{dt} + pW = 0$$

$$\frac{dW}{dt} = -pW$$

$$\frac{dW}{W} = \frac{-p}{dt}$$

$$\ln W = C + -\int_0^t p\, dt$$

$$W = C_1 e^{-\int_0^t p\, dt}$$

true for y_1, y_2 solutions for a diff eq

- $W = 0$ iff $C_1 = 0$. So if $W = 0$ at a point, it must be 0 at every point.
  - Ex: te^t is not a valid Wronskian because it can be 0 or nonzero depending on the value of t.

☆ If $W = 0$, y_1, y_2 are Lin. dependent

$\Leftrightarrow$ If $W \neq 0$ at any point, y_1, y_2 L.I.

Wronskian Lemma

Usage:

Are t^2, te^t solutions to the same diff eq?

$$W = \begin{vmatrix} t^2 & te^t \\ 2t & te^t + e^t \end{vmatrix} = t^2(te^t + e^t) - 2t^2e^t$$

$$= e^t(t^3 - t^2)$$

Since $e^t(t^3 - t^2) = 0$ for $t = 0$ and $\neq 0$ for $t = 3$, t^2 and te^t cannot be solutions to the same diff eq

Rank Theorem: $\text{Rank}(A) + \dim(\text{Null}(A)) = \dim(V)$ for A representing $T: V \to W$

Part 4: More Resources

Adjunct resources: https://drive.google.com/drive/u/2/folders/1LOEowvpxZDp5Y3uzMlKcjMzakU-koTsH

Practice exams: https://campuswire.com/c/GCA89AFCD/feed/5